Eötvös Loránd Tudományegyetem
Bölcsészettudományi Kar

Doktori Disszertáció

Gömöri Márton

The Principle of Relativity

An Empiricist Analysis

(A relativitás elve – empirista elemzés)

Filozófiatudományi Doktori Iskola	
Vezetője:	Boros Gábor, DSc, egyetemi tanár
Logika és Tudományfilozófia Program	
Vezetője:	E. Szabó László, DSc, egyetemi tanár
A bizottság tagjai:	
A bizottség elnöke:	Bodnár István, egyetemi tanár, CSc, egyetemi tanár
Felkért bírálók:	Andréka Hajnal, DSc, professzor
	Szabó Gábor, PhD, tudományos főmunkatárs
A bizottság titkára:	Ambrus Gergely, PhD, egyetemi docens
A bizottság további tagjai:	Máté András, CSc, egyetemi docens
	Mekis Péter, PhD, adjunktus
	Székely Gergely, PhD, tudományos munkatárs
Témavezető:	E. Szabó László, DSc, egyetemi tanár

Budapest, 2015

ADATLAP
a doktori értekezés nyilvánosságra hozatalához

I. A doktori értekezés adatai

A szerző neve: *Gömöri Márton*..
MTMT-azonosító: *10038576*...
A doktori értekezés címe és alcíme: *The principle of relativity - an empiricist analysis*......
DOI-azonosító: *10.15476/ELTE.2015.169*...
A doktori iskola neve: *Filozófiatudományi*...
A doktori iskolán belüli doktori program neve: *Logika és tudományfilozófia*.......................
A témavezető neve és tudományos fokozata: *E. Szabó László, DSc*.......................................
A témavezető munkahelye: *ELTE BTK, Filozófia Intézet, Logika Tanszék*...........................

II. Nyilatkozatok

1. A doktori értekezés szerzőjeként

a) hozzájárulok, hogy a doktori fokozat megszerzését követően a doktori értekezésem és a tézisek nyilvánosságra kerüljenek az ELTE Digitális Intézményi Tudástárban. Felhatalmazom az ELTE BTK Doktori és Tudományszervezési Hivatal ügyintézőjét, Manhercz Mónikát, hogy az értekezést és a téziseket feltöltse az ELTE Digitális Intézményi Tudástárba, és ennek során kitöltse a feltöltéshez szükséges nyilatkozatokat.

b) kérem, hogy a mellékelt kérelemben részletezett szabadalmi, illetőleg oltalmi bejelentés közzétételéig a doktori értekezést ne bocsássák nyilvánosságra az Egyetemi Könyvtárban és az ELTE Digitális Intézményi Tudástárban;

c) kérem, hogy a nemzetbiztonsági okból minősített adatot tartalmazó doktori értekezést a minősítés (dátum)-ig tartó időtartama alatt ne bocsássák nyilvánosságra az Egyetemi Könyvtárban és az ELTE Digitális Intézményi Tudástárban;

d) kérem, hogy a mű kiadására vonatkozó mellékelt kiadó szerződésre tekintettel a doktori értekezést a könyv megjelenéséig ne bocsássák nyilvánosságra az Egyetemi Könyvtárban, és az ELTE Digitális Intézményi Tudástárban csak a könyv bibliográfiai adatait tegyék közzé. Ha a könyv a fokozatszerzést követőn egy évig nem jelenik meg, hozzájárulok, hogy a doktori értekezésem és a tézisek nyilvánosságra kerüljenek az Egyetemi Könyvtárban és az ELTE Digitális Intézményi Tudástárban.

2. A doktori értekezés szerzőjeként kijelentem, hogy

a) az ELTE Digitális Intézményi Tudástárba feltöltendő doktori értekezés és a tézisek saját eredeti, önálló szellemi munkám és legjobb tudomásom szerint nem sértem vele senki szerzői jogait;

b) a doktori értekezés és a tézisek nyomtatott változatai és az elektronikus adathordozón benyújtott tartalmak (szöveg és ábrák) mindenben megegyeznek.

3. A doktori értekezés szerzőjeként hozzájárulok a doktori értekezés és a tézisek szövegének Plágiumkereső adatbázisba helyezéséhez és plágiumellenőrző vizsgálatok lefuttatásához.

Kelt: Budapest, 2015. október 11.

Gömöri Márton

a doktori értekezés szerzőjének aláírása

Contents

I have for long thought that if I had the opportunity to teach this subject, I would emphasize the continuity with earlier ideas. Usually it is the discontinuity which is stressed, the radical break with more primitive notions of space and time. Often the result is to destroy completely the confidence of the student in perfectly sound and useful concepts already acquired. (From J. S. Bell: “How to teach special relativity”, Bell 1987, p. 67.)

Chapter 1

Introduction

1. Here is Galileo's lively description of the special principle of relativity as appearing in the famous passages of the *Dialogue*:

> Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though doubtless when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than toward the prow even though the ship is moving quite rapidly, despite the fact that during the time that you are in the air the floor under you will be going in a direction opposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite. The droplets will fall as before into the vessel beneath without dropping toward the stern, although while the

> drops are in the air the ship runs many spans. The fish in their water will swim toward the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air. And if smoke is made by burning some incense, it will be seen going up in the form of a little cloud, remaining still and moving no more toward one side than the other. The cause of all these correspondences of effects is the fact that the ship's motion is common to all the things contained in it, and to the air also. That is why I said you should be below decks; for if this took place above in the open air, which would not follow the course of the ship, more or less noticeable differences would be seen in some of the effects noted. (Galilei 1953, pp. 186–187)

Compare Galileo's description with a sample from the huge variety of formulations of the principle available in the literature:

> ... the laws of physical phenomena should be the same, whether for an observer fixed, or for an observer carried along in a uniform movement of translation. (Poincaré 1956, p. 167)

> If a system of coordinates K is chosen so that, in relation to it, physical laws hold good in their simplest form, the same laws hold good in relation to any other system of coordinates K' moving in uniform translation relatively to K. (Einstein 1916, p. 111)

> ... it is impossible to measure or detect the unaccelerated translatory motion of a system through free space or through any ether-like medium. (Tolman 1949, p. 12)

> ... all physical phenomena should have the same course of development in all system of inertia, and observers installed in different systems of inertia should thus as a result of their experiments arrive at the establishment of the same laws of nature. (Møller 1955, p. 4)

> ... the laws of Physics take the same mathematical form in all inertial frames (Sardesai 2004, p. 1)

> The same laws of nature are true for all inertial observers. (Madarász 2002, p. 84)

The uniform translatory motion of any system can not be detected by an observer traveling with the system and making observations on it alone. (Comstock 1909, p. 767)

The laws of nature and the results of all experiments performed in a given frame of reference are independent of the translational motion of the system as a whole. More precisely, there exists a ... set of equivalent Euclidean reference frames . . . in which all physical phenomena occur in an identical manner. (Jackson 1999, p. 517)

If we express some law of physics using the quantities of one inertial frame of reference, the resulting statement of the law will be exactly the same in any other inertial frame of reference. ... we write down exactly the same sentence to express the law in each inertial frame. (Norton 2013)

... all inertial frames are equivalent for the performance of all physical experiments (Rindler 2006, p. 12)

... the laws of physics are invariant under a change of inertial coordinate system (Ibid., p. 40)

The outcome of any physical experiment is the same when performed with identical initial conditions relative to any inertial coordinate system. (Ibid.)

... all the laws of nature are identical in all inertial systems of reference. In other words, the equations expressing the laws of nature are invariant with respect to transformations of coordinates and time from one inertial system to another. This means that the equation describing any law of nature, when written in terms of coordinates and time in different inertial reference systems, has one and the same form. (Landau and Lifshitz 1980, p. 1)

... the phenomena in a given reference system are, in principle, independent of the translational motion of the system as a whole. To put it more precisely: there exists a triply infinite set of reference systems moving rectilinearly and uniformly relativity to one another, in which the phenomena occur in an identical manner. (Pauli 1958, p. 4)

The postulate of relativity implies that a uniform motion of the center of mass of the universe relativity to a closed system will be without influence on the phenomena in such a system. (Ibid., p. 5)

> ... experience teaches us that ... all laws of physical nature which have been formulated with reference to a definite coordinate system are valid, in precisely the same form, when referred to another co-ordinate system which is in uniform rectilinear motion with respect to the first. ... All physical events take place in any system in just the same way, whether the system is at rest or whether it is moving uniformly and rectilinearly. (Schlick 1920, p. 10)
>
> ... laws must be Lorentz covariant. Lorentz covariance became synonymous with satisfaction of the principle of relativity. (Norton 1993, p. 796)
>
> The laws of physics don't change, even for objects moving in inertial (constant speed) frames of reference. (Zimmerman Jones and Robbins 2009, p. 84)
>
> ... the basic physical laws are the invariant relationships, the same for all observers (Bohm 1996, p. viii)
>
> ... laws of physics must satisfy the requirement of being relationships of the same form, in every frame of reference (Ibid., p. 54).

While there is a longstanding discussion about the interpretation of the extended, general principle of relativity, there seems to be a consensus that the above quotations express an absolutely clear statement.

This statement plays a fundamental role in modern physics from various perspectives. As a basic premise of Einstein's special relativity, the principle of relativity[1] is an indispensable ingredient of physics within the relativistic domain, in particular, high energy particle physics. Conceived as a meta-law, the principle provides a powerful and restrictive tool for constructing new theories. The relativity principle is often put in elegant, geometric terms in order to emphasize that the principle expresses a fundamental symmetry of space-time. This kind of symmetry plays an essential role from a metaphysical point of view: the essential physical attributes, like mass or spin, of elementary physical systems are determined by how the physical states of the system in question transform under these symmetry transformations provided by the relativity principle.

But where does this outstanding efficiency of the principle of relativity come from? What facts of the physical world does the principle exactly captures? And how is it possible that such a loosely, sometimes even metaphorically, formulated sentence can do so much work in physics?

[1] From now on the "principle of relativity" will always mean the special principle.

The present paper challenges the received sentiments on the special principle of relativity. It will be argued that the relativity principle is not equivalent with Lorentz symmetry/covariance—which is usually taken to be synonymous with the principle; that the relativity principle, in its most general sense, has an unambiguous meaning; and in fact its satisfaction is only provided under special physical conditions.

2. The dissertation is structured as follows. Chapter 2 is concerned with the *meaning* of the relativity principle. My aim is to develop a precise language in order to provide a precise formulation of the principle. A mathematical language will be constructed in terms of which the statement of the relativity principle will be properly formulated. The benefit of the formal reconstruction is that it makes explicit all the necessary conceptual components of the relativity principle. It will be proved, in terms of this precise language, that the relativity principle is logically independent from the requirement of Lorentz symmetry/covariance.

Chapter 3 and 4 are concerned with the question of whether the principle of relativity is *true*. Despite that the principle is often considered as a meta-law, it will be pointed out that ultimately this is an empirical question. However, it is a natural idea to apply what J. S. Bell (1987) calls "Lorentzian pedagogy" according to which "the laws of physics in any one reference frame account for all physical phenomena, including the observations of moving observers", that is, including whether the relativity principle is true or not. The "Lorentzian pedagogy" will be applied to decide whether the relativity principle holds in classical electrodynamics, in the paradigmatic case of a relativistic theory. Interestingly, this analysis has never been carefully performed.

The answer can be given by the laws of electrodynamics only if the question is properly formulated. We must be able to tell what the measurement operations by means of which a moving observer determines the values of electrodynamic quantities exactly are. As it turns out, to give the precise, non-circular operational definitions of the electrodynamics quantities is a non-trivial problem. A possible construction will be proposed in Chapter 3.

The results of the analysis are the following. By applying the "Lorentzian pedagogy", independently of the relativity principle, it will be shown in Chapter 3 that the transformation rules of the electrodynamic quantities are identical with the ones usually obtained in the literature by presuming the covariance of the equations of electrodynamics, and that the covariance is indeed satisfied. As for the relativity principle, the situation is more complex. As is shown in Chapter 2, the covariance of physical equations is not enough for the principle; whether the principle of relativity holds depends on the details of the solutions describing moving objects. This raises the question how to understand the concept of "moving object" in general. Our surprising conclusion will be that this question has no satisfactory answer; "motion" is a vague concept. It will be argued in Chapter 4 that in case of a general electrodynamic system there seems no rational way to understand the concept of the "electromagnetic field in motion". This suggests, contrary to what is believed, that the principle of relativity does not even have an unambiguous meaning in classical electrodynamics.

Chapter 2

What Exactly Does the Principle of Relativity Say?

2.1 Einstein 1905

3. Here is how Einstein famously begins his 1905 paper on the special theory of relativity:

> It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise—assuming equality of relative motion in the two cases discussed—to electric currents of the same path and intensity as those produced by the electric forces in the former case.
>
> Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium," suggest that the phenomena of electrodynamics as well as of mechanics possess

> no properties corresponding to the idea of absolute rest. They suggest rather that, as has already been shown to the first order of small quantities, the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate [...]. (Einstein 1905, pp. 37–38)

It is well known that the magnet-conductor thought experiment played a decisive role in leading Einstein to postulate the relativity principle; in fact, it was much more significant for him than the empirical findings of Michelson and Morley also referred to in the above passage (Norton 2004, pp. 48–50). Considering this fact, it is remarkable to notice that Einstein's observation regarding the phenomenological equivalence of the magnet-conductor scenarios is only valid in an approximate sense. Take Maxwell's electrodynamics in a given inertial frame of reference and let us see how it accounts for the two cases. Of course, whether and how a steady state current ensues in the material of a conductor is a complex matter involving details of the interaction of particles within the material (cf. Bandyopadhyay 2012, Chaper 1; Scanlon, Henriksen and Allen 1969, pp. 701–703). What is important here is that the current will depend on the Lorentz force exerted by external fields on the charges in the conductor. Pick one particle of charge q and calculate the force on it in the two cases. If the magnet is at rest and the conductor is in motion with velocity $\mathbf{V}$, the force on the charge moving together with the conductor[1] is

$$\mathbf{F} = q\mathbf{V} \times \mathbf{B} \tag{2.1}$$

where $\mathbf{B}(\mathbf{r})$ is a time-independent, spatially inhomogeneous magnetic field determined by the shape of the magnet at rest. If the conductor is at rest and the magnet is in motion with velocity $-\mathbf{V}$, the force on the stationary charge is $\mathbf{F} = q\mathbf{E}$, where the electric field $\mathbf{E}(\mathbf{r}, t)$ is induced by the changing magnetic field of the moving magnet according to Faraday's law

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \tilde{\mathbf{B}}(\mathbf{r}, t)}{\partial t} \tag{2.2}$$

Here $\tilde{\mathbf{B}}(\mathbf{r}, t)$ is the field of the magnet when it is in motion with velocity $-\mathbf{V}$.

[1] It is assumed that the charge is at rest relative to the conductor. Nevertheless, if current ensues in the conductor due to the force by the external field, this gives an additional drift velocity $\mathbf{v}$ to the charge relative to the conductor. It is easy to see, however, that $\mathbf{v}$ is going to be perpendicular to $\mathbf{B}$, so it does not contribute to the force on the charge. (Cf. Giuliani 2008)

Assuming that the field travels together with the magnet in the sense of

$$\tilde{\mathbf{B}}\left(\mathbf{r},t\right)=\mathbf{B}\left(\mathbf{r}+\mathbf{V}t\right) \tag{2.3}$$

one can show that Faraday's law has the solution $\mathbf{E}\left(\mathbf{r}\right)=\mathbf{V}\times\mathbf{B}\left(\mathbf{r}\right)$. This yields exactly the same expression for the force on the charge (2.1) as in the previous case.

Notice, however, that (2.3) can only be true in an approximate sense. It is not straightforward to calculate directly in the given frame of reference how the field of a moving magnet is going to look like. Nevertheless, taking into account the relativistic effects predicted by Einstein's special relativity itself, one should expect that the magnet together with its field will suffer "relativistic deformations" as set into motion, and hence factor $(1-V^2/c^2)^{-\frac{1}{2}}$ will enter formula (2.3).[2] $(1-V^2/c^2)^{-\frac{1}{2}}$ is going to appear in the solution of Faraday's law and hence in the expression of the force on the charge in case when the magnet is in motion; whereas no additional factor shows up in the force when the charge is in motion. Only in the non-relativistic limit of $V/c\to 0$ one can expect that the "observable phenomenon depends only on the relative motion of the conductor and the magnet".

4. This should be the case when the conductor + magnet system in *two different* states of overall motion is compared from *one single* frame of reference. A slightly different reading of Einstein's example appears in many textbooks. For example, here is how Griffiths 1999 puts it:

> Suppose we mount a wire loop on a freight car, and have the train pass between the poles of a giant magnet [...]. As the loop rides through the magnetic field, a motional emf [electromotive force] is established; according to the flux rule [...],
>
> $$\mathcal{E}=-\frac{d\phi}{dt} \tag{2.4}$$
>
> This emf [...] is due to the magnetic force on charges in the wire loop, which are moving along with the train. On the other hand, if someone on the train naïvely applied the laws of electrodynamics in that system, what would the prediction be? No magnetic force, because the loop is at rest. But as the magnet flies by, the magnetic field in the freight car will change, and a changing magnetic field induces an electric field, by Faraday's law. The resulting electric force would generate an emf in the

[2] Compare with the expressions of the electromagnetic field of the static versus uniformly moving point charge (2.76) and (2.77) below.

> loop given by [...]:
>
> $$\mathcal{E} = -\frac{d\phi}{dt} \tag{2.5}$$
>
> Because Faraday's law and the flux rule predict exactly the same emf, people on the train will get the right answer, even though their physical interpretation of the process is completely wrong.
>
> Or is it? Einstein could not believe this was a mere coincidence; he took it, rather, as a clue that electromagnetic phenomena, like mechanical ones, obey the principle of relativity. (Griffiths 1999, p. 478)

In this wording, *one single* phenomenon is described as seen from *two different* frames of reference. It easy to show, however, that the answers of the two observers will only coincide in a non-relativistic approximation again. The electromotive force is defined as

$$\mathcal{E} = \oint_C (\mathbf{E} + \mathbf{v} \times \mathbf{B})\, d\mathbf{l} \tag{2.6}$$

where C is the instantaneous contour of the wire loop, and $\mathbf{v}$ is the velocity of the charge located at a given point of C. The emf can be interpreted as the work per unit charge done by the external field along the wire loop at a given instant. Let us now compare the values of $\mathcal{E}$ directly calculated from this definition as accounted for by the two observers.[3] In the magnet frame, again, there is no electric field, and the charges in the wire are moving with velocity $\mathbf{V}$ so

$$\mathcal{E} = \oint_C (\mathbf{V} \times \mathbf{B})\, d\mathbf{l} \tag{2.7}$$

In the conductor frame the charges are at rest so we have

$$\mathcal{E}' = \oint_{C'} \mathbf{E}' d\mathbf{l}' \tag{2.8}$$

One can express $\mathcal{E}'$ in terms of the magnet frame quantities by means of the relativistic transformation rules. As for the electric field strength, $\mathbf{E}' = \frac{\mathbf{V}\times\mathbf{B}}{\sqrt{1-V^2/c^2}}$. Choose a coordinate system in both frames in which the relative velocity of the magnet and conductor points in the x direction. Due to the Lorentz transformation of space and time coordinates $d\mathbf{l}' = \left(\frac{dl_x}{\sqrt{1-V^2/c^2}}, dl_y, dl_z\right)$; but since $\mathbf{V} \times \mathbf{B}$ is perpendicular to $\mathbf{V}$ we have $\mathbf{E}'d\mathbf{l}' = \frac{\mathbf{V}\times\mathbf{B}}{\sqrt{1-V^2/c^2}} d\mathbf{l}$ for the integrand. Finally, C' is the *instantaneous* contour of the wire loop in its rest frame. The relativity of simultaneity implies that

[3] One would obtain the same result by comparing the rate of change of magnetic flux $\frac{d\phi}{dt}$ in the two frames.

this is not the same set of events in space-time as C. Therefore, integrals (2.7) and (2.8), and hence $(\mathbf{V} \times \mathbf{B})\, d\mathbf{l}$, have to be evaluated along different curves in space-time. Nevertheless, as $\mathbf{B}$ is constant in time in the magnet frame, a little reflection shows that

$$\oint_{C'} (\mathbf{V} \times \mathbf{B})\, d\mathbf{l}' = \oint_{C} (\mathbf{V} \times \mathbf{B})\, d\mathbf{l} \tag{2.9}$$

All in all, we have $\mathcal{E}' = \frac{\mathcal{E}}{\sqrt{1-V^2/c^2}}$. Again, only in the non-relativistic limit will the predictions of the two observers coincide.

5. It is the relativistic effects and transformation laws derived by Einstein himself in the 1905 paper that render his observations on the magnet-conductor case, by which he motivates the relativity principle, invalid. Is Einstein's relativity thoroughly inconsistent? Or should the principle be understood in a different way?

Many hold it should. Here is a typical passage from the literature, voiced in Scanlon, Henriksen and Allen 1969, in which the authors, in an almost off-handed manner, comment on the relativistic aspects of Faraday's law of induction:

> [I]t is clear that Eqs. (2.2) and (2.4) [numbers refer to the present text] will be covariant together when $\mathcal{E}$ is the Faraday emf. But of course $\mathcal{E}$ is not an invariant number so that the question arises as to whether it is a physically significant quantity for any given observer. (Scanlon, Henriksen and Allen 1969, p. 701)

The authors approve that "$\mathcal{E}$ is not an invariant number", that is different observers obtain different values for the electromotive force around a circuit. Yet they do not even bother to reflect on whether this fact may cause any trouble for relativity, considering Einstein's magnet-conductor example. This is because they take it as obvious that what the relativity principle actually requires is not that physical quantities should have the same values in different frames (in a given physical situation), but that they should vary together so that the functional relationships among them remain the same in all inertial frames of reference. In other words, the physical equations which these quantities obey, among them Faraday's law (2.2) and the flux rule (2.4), must be *covariant*.

6. This reading of the relativity principle also finds support in the 1905 paper. This is how Einstein applies the principle when he derives the transformation laws of the electromagnetic field strengths:

Let the Maxwell–Hertz equations for empty space hold good for the stationary system K, so that we have

$$\frac{1}{c^2}\frac{\partial E_x}{\partial t} = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \tag{2.10}$$

$$\frac{1}{c^2}\frac{\partial E_y}{\partial t} = \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \tag{2.11}$$

$$\frac{1}{c^2}\frac{\partial E_y}{\partial t} = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \tag{2.12}$$

$$\frac{\partial B_x}{\partial t} = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \tag{2.13}$$

$$\frac{\partial B_y}{\partial t} = \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \tag{2.14}$$

$$\frac{\partial B_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \tag{2.15}$$

[notations and usage of units follow the present text] [...]

If we apply to these equations the transformation developed in § 3 [transformation of space and time coordinates; see (4.39)–(4.42)], by referring the electromagnetic processes to the system of co-ordinates there introduced, moving with the velocity V, we obtain the equations

$$\frac{1}{c^2}\frac{\partial E_x}{\partial t'} = \frac{\partial}{\partial y'}\left(\frac{B_z - \frac{V}{c^2}E_y}{\sqrt{1-\frac{V^2}{c^2}}}\right) - \frac{\partial}{\partial z'}\left(\frac{B_y + \frac{V}{c^2}E_z}{\sqrt{1-\frac{V^2}{c^2}}}\right) \tag{2.16}$$

$$\frac{1}{c^2}\frac{\partial}{\partial t'}\left(\frac{E_y - VB_z}{\sqrt{1-\frac{V^2}{c^2}}}\right) = \frac{\partial B_x}{\partial z'} - \frac{\partial}{\partial x'}\left(\frac{B_z - \frac{V}{c^2}E_y}{\sqrt{1-\frac{V^2}{c^2}}}\right) \tag{2.17}$$

$$\frac{1}{c^2}\frac{\partial}{\partial t'}\left(\frac{E_z + VB_y}{\sqrt{1-\frac{V^2}{c^2}}}\right) = \frac{\partial}{\partial x'}\left(\frac{B_y + \frac{V}{c^2}E_z}{\sqrt{1-\frac{V^2}{c^2}}}\right) - \frac{\partial B_x}{\partial y'} \tag{2.18}$$

$$\frac{\partial B_x}{\partial t'} = \frac{\partial}{\partial z'}\left(\frac{E_y - VB_z}{\sqrt{1-\frac{V^2}{c^2}}}\right) - \frac{\partial}{\partial y'}\left(\frac{E_z + VB_y}{\sqrt{1-\frac{V^2}{c^2}}}\right) \tag{2.19}$$

$$\frac{\partial}{\partial t'}\left(\frac{B_y + \frac{V}{c^2}E_z}{\sqrt{1-\frac{V^2}{c^2}}}\right) = \frac{\partial}{\partial x'}\left(\frac{E_z + VB_y}{\sqrt{1-\frac{V^2}{c^2}}}\right) - \frac{\partial E_x}{\partial z'} \tag{2.20}$$

$$\frac{\partial}{\partial t'}\left(\frac{B_z - \frac{V}{c^2}E_y}{\sqrt{1-\frac{V^2}{c^2}}}\right) = \frac{\partial E_x}{\partial y'} - \frac{\partial}{\partial x'}\left(\frac{E_y - VB_z}{\sqrt{1-\frac{V^2}{c^2}}}\right) \tag{2.21}$$

Now the principle of relativity requires that if the Maxwell–Hertz equations for empty space hold good in system K, they also hold good in system K'; that is to say that the vectors of the electric and the magnetic force—$\left(E'_x, E'_y, E'_z\right)$ and $\left(B'_x, B'_y, B'_z\right)$—of the moving system K', which are defined by their ponderomotive effects on electric or magnetic masses respectively, satisfy the following equations:—

$$\frac{1}{c^2}\frac{\partial E'_x}{\partial t'} = \frac{\partial B'_z}{\partial y'} - \frac{\partial B'_y}{\partial z'} \tag{2.22}$$

$$\frac{1}{c^2}\frac{\partial E'_y}{\partial t'} = \frac{\partial B'_x}{\partial z'} - \frac{\partial B'_z}{\partial x'} \tag{2.23}$$

$$\frac{1}{c^2}\frac{\partial E'_y}{\partial t'} = \frac{\partial B'_y}{\partial x'} - \frac{\partial B'_x}{\partial y'} \tag{2.24}$$

$$\frac{\partial B'_x}{\partial t'} = \frac{\partial E'_y}{\partial z'} - \frac{\partial E'_z}{\partial y'} \tag{2.25}$$

$$\frac{\partial B'_y}{\partial t'} = \frac{\partial E'_z}{\partial x'} - \frac{\partial E'_x}{\partial z'} \tag{2.26}$$

$$\frac{\partial B'_z}{\partial t'} = \frac{\partial E'_x}{\partial y'} - \frac{\partial E'_y}{\partial x'} \tag{2.27}$$

Evidently the two systems of equations found for system K' must express exactly the same thing, since both systems of equations are equivalent to the Maxwell–Hertz equations for system K. Since, further, the equations of the two systems agree, with the exception of the symbols for the vectors [of electric and magnetic force], it follows that the functions occurring in the systems of equations at corresponding places must agree [...]. Thus we have the relations [...]

$$E'_x = E_x \tag{2.28}$$

$$E'_y = \frac{E_y - VB_z}{\sqrt{1-\frac{V^2}{c^2}}} \tag{2.29}$$

$$E'_z = \frac{E_z + VB_y}{\sqrt{1-\frac{V^2}{c^2}}} \tag{2.30}$$

$$B'_x = B_x \tag{2.31}$$

$$B'_y = \frac{B_y + \frac{V}{c^2}E_z}{\sqrt{1-\frac{V^2}{c^2}}} \tag{2.32}$$

$$B'_z = \frac{B_z - \frac{V}{c^2}E_y}{\sqrt{1-\frac{V^2}{c^2}}} \tag{2.33}$$

(Einstein 1905, pp. 51–53)

I deliberately quoted this passage in detail because it brings out important subtleties. Einstein clearly formulates what he takes to be the requirement of the relativity principle in this case:

1) The primed quantities of frame K' must *obey* equations of the same form as equations (2.10)–(2.15) the unprimed quantities of frame K obey; hence (2.22)–(2.27).

2) The equations which the primed quantities obey, (2.22)–(2.27), must *express the same thing* as the equations satisfied by the unprimed quantities, (2.10)–(2.15); that is, the primed equations must be obtainable by expressing the unprimed ones in terms of the primed quantities through their transformation laws.

The first condition would not be enough in order to arrive at the transformation laws of the field strengths in a way Einstein does. For what makes it possible to simply read off the transformations by comparing the structure of (2.16)–(2.21) and (2.22)–(2.27) is that these two sets of equations, on this account of the relativity principle, must *carry the same physical content*; otherwise nothing would guarantee that "the functions occurring in the systems of equations at corresponding places must agree" for completely unrelated equations with completely unrelated quantities in them may also share the same algebraic/typographic structure. We are not entitled to identify, say, the pressure in a fluid with the electric field strength in a light ray just because both of them obey a wave equation.

But do equations (2.16)–(2.21) and (2.22)–(2.27) indeed carry the same physical content? Notice that while Einstein talks about the Maxwell–Hertz equations, he actually writes down only two of them, the Ampère–Maxwell and Faraday's laws (in vacuum). However, when expressing these equations in terms of the primed space and time coordinates he tacitly also makes use of the two remaining Maxwell equations

$$\nabla \cdot \mathbf{E} = 0 \tag{2.34}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.35}$$

—it is easy to check that without them one cannot arrive at (2.16)–(2.21) from (2.10)–(2.15). This means that (2.16)–(2.21) is in fact *not* equivalent to (2.10)–(2.15); whereas (2.22)–(2.27), according to the second requirement of the relativity principle above, must be. Therefore, (2.16)–(2.21) and (2.22)–(2.27) do *not* express the same thing. How are then we entitled to match the primed and unprimed field strengths merely on the basis of their algebraic/typographic roles in these *physically inequivalent* equations?

To put it slightly differently, consider the transformation laws (2.28)–(2.33) Einstein derives. An easy calculation shows that the Ampère–Maxwell and Faraday's laws (2.10)–(2.15) are *not* covariant against these transformations; expressing them in terms of the primed quantities through (2.28)–(2.33) leads to more complicated equations then (2.22)–(2.27). Only when taking into account what obtains as a result of transforming the two other Maxwell equations (2.34)–(2.35) into primed variables too, does one arrive at (2.22)–(2.27). It is the *whole* system of Maxwell's equations that is covariant under the transformations Einstein derives, but not the Ampère–Maxwell or Faraday's law separately.[4]

7. The same applies to the flux rule (2.4) the covariance of which Scanlon, Henriksen and Allen 1969 refers to in the quoted passage. The flux rule reads

$$\oint_C (\mathbf{E} + \mathbf{v} \times \mathbf{B})\, d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} d\mathbf{a} \tag{2.36}$$

where the right hand side is the rate of change of magnetic flux through a surface S whose boundary is the contour C of the wire loop. Note that everything in this formula is taken at a given instant of time, relative to a given fame of reference. Now write down the flux rule in the primed variables of a different frame:

$$\oint_{C'} (\mathbf{E}' + \mathbf{v}' \times \mathbf{B}')\, d\mathbf{l}' = -\frac{d}{dt'}\int_{S'} \mathbf{B}' d\mathbf{a}' \tag{2.37}$$

Due to the relativity of simultaneity, as mentioned earlier, this equation will carve out events in space-time different from the ones (2.36) concerns; in the sense that the two equations put constraints on the values of the field strengths taken in *different* regions of space-time. Thus, the two equations do *not* express the same thing; they express *different* facts about the behavior of the electromagnetic field. Therefore, one cannot obtain (2.37) by simply expressing (2.36) in terms of the primed quantities through the transformation laws. The flux rule is *not* covariant. Yet as the flux rule follows from Maxwell's equations,[5] what is indeed true is that the field strengths will *obey* the flux rule in any given frame of reference so long as Maxwell's equations

[4] In fact, as it will be shown in Section 3.7, (2.10)–(2.12) + (2.34) and (2.13)–(2.15) + (2.35) constitute two covariant sets of equations separately.

[5] The flux rule can be derived from (2.13)–(2.15) and (2.35) (Scanlon, Henriksen and Allen 1969, p. 701). If one is to interpret the electromotive force as the work on charges in the wire then the Lorentz force law is also needed to be invoked.

are respected in that frame.

8. If the relativity principle means covariance, *whose* covariance should it then be taken to be? What is so special about the whole covariant system of Maxwell's equations, as opposed to other non-covariant equations, with regard to what the relativity principle is meant to say? Or should the principle be relaxed to the requirement of what I phrased in Paragraph **6** as Einstein's first condition demanding that equations of the same form must *hold good* in all inertial frames *regardless of what they express*? Is it a sensible requirement? Does it make sense to require that, say, Gauss's law of electrostatics in one frame must have the same form as Lorentz's force law in another? Moreover, many physical equations fail to satisfy even this weaker condition; nevertheless, the principle of relativity just as well seems to apply to them, as the following will demonstrate.

9. For consider how Einstein applies the principle when he derives the equation of motion for the moving point charge in the closing section of the 1905 paper:

> Let there be in motion in an electromagnetic field an electrically charged particle (in the sequel called an "electron"), for the law of motion of which we assume as follows:—
>
> If the electron is at rest at a given epoch, the motion of the electron ensues in the next instant of time according to the equations
>
> $$m\frac{d^2x}{dt^2} = \epsilon E_x \tag{2.38}$$
>
> $$m\frac{d^2y}{dt^2} = \epsilon E_y \tag{2.39}$$
>
> $$m\frac{d^2z}{dt^2} = \epsilon E_z \tag{2.40}$$
>
> [...]
>
> Now, secondly, let the velocity of the electron at a given epoch be V. We seek the law of motion of the electron in the immediately ensuing instants of time.
>
> Without affecting the general character of our considerations, we may and will assume that the electron, at the moment when we give it our attention, is at the origin of the co-ordinates, and moves with the velocity V along the axis of x of the system K. It is then clear that at the given moment ($t = 0$) the electron is at rest relatively to a system of co-ordinates which is in parallel motion with velocity V along the axis of

x.

From the above assumption, in combination with the principle of relativity, it is clear that in the immediately ensuing time (for small values of t) the electron, viewed from the system K', moves in accordance with the equations

$$m\frac{d^2x'}{dt'^2} = \epsilon E'_x \tag{2.41}$$

$$m\frac{d^2y'}{dt'^2} = \epsilon E'_y \tag{2.42}$$

$$m\frac{d^2z'}{dt'^2} = \epsilon E'_z \tag{2.43}$$

[...] If, further, we decide that when $t = x = y = z = 0$ then $t' = x' = y' = z' = 0$, the transformation equations of §§ 3 and 6 [transformations of space and time coordinates (4.39)–(4.42), and transformation of field strengths (2.28)–(2.33)] hold good [...].

With the help of these equations we transform the above equations of motion from system K' to system K, and obtain

$$\frac{d^2x}{dt^2} = \frac{\epsilon}{m}\left(1-\frac{V^2}{c^2}\right)^{\frac{3}{2}} E_x \tag{2.44}$$

$$\frac{d^2y}{dt^2} = \frac{\epsilon}{m}\sqrt{1-\frac{V^2}{c^2}}\,(E_y - VB_z) \tag{2.45}$$

$$\frac{d^2z}{dt^2} = \frac{\epsilon}{m}\sqrt{1-\frac{V^2}{c^2}}\,(E_z + VB_y) \tag{2.46}$$

(Einstein 1905, pp. 61–62)

It is clear that equations (2.38)–(2.40) are *not* covariant. Indeed, (2.38)–(2.40) and its primed version (2.41)–(2.43) do *not* express the same thing: they characterize the behavior of the charge in *different* states of motion, relative to any given frame. Nor do these two sets of equations even *hold good* at the same time, that is in a given physical situation, for the charge can only take up one single state of motion at a time. Apparently, neither of these conditions is what Einstein takes to be the requirement of the relativity principle in this case.

Einstein compares *two* situations: one in which the charge is at rest relative to K and another one in which it is in motion with velocity V relative to K, co-moving with K'. He takes the relativity principle to say that the equations describing the second situation expressed in terms of frame K' must have the same form as the equations describing the first situation expressed in terms of frame K. That is, the behaviors of the charge *in the two different states of motion* must look the same as described from the *co-moving* frames. Note that this condition does not seem to follow even from the covariance of the full-fledged Lorentz equation of the particle; for there is no way to refer to the specific situations involved only in terms of the covariance of a single equation.

10. It is important to observe that Einstein specifies the two situations only in terms of how the charge behaves—being at rest versus being in motion—, but says nothing about the electromagnetic field. It must be clear, however, that the relativity principle is not supposed to concern the behavior of the charge alone, in separation from its environment with which it interacts, but that of the *whole* coupled system under consideration, that is, the equations describing the charge and the electromagnetic field together. To see the significance of this, compare the moving charge case with a simple non-relativistic system consisting of a box on a table. If the box is at rest, its equation of motion is of the form

$$m\frac{d^2\mathbf{r}}{dt^2} = 0 \tag{2.47}$$

If the box is in motion with velocity $\mathbf{V}$ in the plane of the tabletop being at rest, its equation of motion is

$$m\frac{d^2\mathbf{r}}{dt^2} = -\mu\mathbf{V} \tag{2.48}$$

due to friction on the table; where μ is the coefficient of friction. Now express (2.48) through the Galilean transformations in terms of the primed quantities associated with the frame of reference co-moving with the box at the initial instant:

$$m'\frac{d^2\mathbf{r}'}{dt^2} = -\mu'\mathbf{V} \tag{2.49}$$

This is the description of the moving box in the co-moving frame. In analogy with Einstein's (2.41)–(2.43), this equation should be of the same form as (2.47) on account of the relativity principle as applied by Einstein for the moving charge case. But it is manifestly not. However, it is certainly not supposed to count as a violation of the principle if the system in question is "disrupted", part of it being set in motion, part of it being kept at rest. Only when the *whole* system, the box and the table together, is set into motion, only then are the equations describing the box (and the table) in the co-moving frame going to look the same as the original equations in the rest frame. It seems to be an additional feature of Lorentz's equation that Einstein requires when demanding that equations (2.41)–(2.43) should hold for the moving charge regardless of what the electric field strength is. This condition, while may well be true for some cases, need not in general seems to follow from what the principle of relativity, as it is understood by Einstein in the moving charge case, is meant to imply (and, again, not even from the covariance of the Lorentz equation). Only when the particle + field system *as a whole*, including the electromagnetic field together with its sources, set into two different states of overall motion is compared, only then should the relativity principle require their respective descriptions in the co-moving frames to have the same form.

11. Had Einstein applied this version of the principle to the magnet-conductor case, he would have realized that the observation he makes in the beginning of the paper is not correct. For consider again the magnet + conductor system in two different states of overall motion: one in which the magnet is at rest and the conductor is in motion with velocity $\mathbf{V} = (V, 0, 0)$, and another one in which the conductor is at rest and the magnet is in motion with velocity $-\mathbf{V}$, relative to an inertial frame K. In the first case, again, the magnetic field of the magnet is $\mathbf{B}(\mathbf{r})$ and there is no electric field—relative to K. Now take an inertial frame K' being in motion with velocity $-\mathbf{V}$ relative to K, co-moving with the magnet in the second scenario. According to the latter account of the relativity principle, in analogy with Einstein's requirement (2.41)–(2.43) versus (2.38)–(2.40), the description of the electromagnetic field in the second case expressed in terms of frame K' must have the same form as that in the first case expressed in terms of frame K: the magnetic field has the functional form $\mathbf{B}(\mathbf{r})$ and no electric field—again, relative to K'. Now transform this field back to frame K by means of the transformation laws of the kinematic and electromagnetic quantities Einstein derives to obtain:

$$\tilde{E}_x(x,y,z,t) = 0 \tag{2.50}$$

$$\tilde{E}_y(x,y,z,t) = -\frac{VB_z(X,y,z)}{\sqrt{1-\frac{V^2}{c^2}}} \tag{2.51}$$

$$\tilde{E}_z(x,y,z,t) = \frac{VB_y(X,y,z)}{\sqrt{1-\frac{V^2}{c^2}}} \tag{2.52}$$

$$\tilde{B}_x(x,y,z,t) = B_x(X,y,z) \tag{2.53}$$

$$\tilde{B}_y(x,y,z,t) = \frac{B_y(X,y,z)}{\sqrt{1-\frac{V^2}{c^2}}} \tag{2.54}$$

$$\tilde{B}_z(x,y,z,t) = \frac{B_z(X,y,z)}{\sqrt{1-\frac{V^2}{c^2}}} \tag{2.55}$$

where $X = \frac{x+Vt}{\sqrt{1-V^2/c^2}}$. This is the electromagnetic field of the moving magnet in frame K. (2.53)–(2.55) shows that (2.3), under the present reading of the relativity principle, is indeed not the correct expression for the magnetic field of the moving magnet in the relativistic domain. Combining (2.50)–(2.52) with the Lorentz force formula one receives $\mathbf{F} = \frac{q\mathbf{V}\times\mathbf{B}}{\sqrt{1-V^2/c^2}}$ for the force on the charge in the conductor when the magnet is in motion, which indeed differs from the expression of the force (2.1) calculated for the case when the conductor is in motion. Notice that it is the relativity principle itself from which this result derives—it is the relativity principle in the fashion as Einstein applies it to the moving charge case plus in the form of the transformation laws he derives from the requirement of covariance. *It is the principle of relativity itself as applied by Einstein in the 1905 paper that is in contradiction with the way he supposed the principle reveals in the magnet-conductor thought experiment.*

12. It is worth tracking how Einstein continues his considerations on the dynamics of the moving charge:

> Taking the ordinary point of view we now inquire as to the "longitudinal" and the "transverse" mass of the moving electron. We write the equations (2.44)–(2.46) in the form
>
> $$\frac{m}{\left(1-\frac{V^2}{c^2}\right)^{\frac{3}{2}}}\frac{d^2x}{dt^2} = \epsilon E_x = \epsilon E'_x \tag{2.56}$$
>
> $$\frac{m}{1-\frac{V^2}{c^2}}\frac{d^2y}{dt^2} = \epsilon\frac{E_y - VB_z}{\sqrt{1-\frac{V^2}{c^2}}} = \epsilon E'_y \tag{2.57}$$
>
> $$\frac{m}{1-\frac{V^2}{c^2}}\frac{d^2z}{dt^2} = \epsilon\frac{E_z + VB_y}{\sqrt{1-\frac{V^2}{c^2}}} = \epsilon E'_z \tag{2.58}$$
>
> and remark firstly that $\epsilon E'_x, \epsilon E'_y, \epsilon E'_z$ are the components of the ponderomotive force acting upon the electron, and are so indeed as viewed in a system moving at the moment with the electron, with the same velocity as the electron. (This force might be measured, for example, by a spring balance at rest in the last-mentioned system.) Now if we call this force simply "the force acting upon the electron," and maintain the equation—mass × acceleration = force—and if we also decide that the accelerations are to be measured in the stationary system K, we derive

from the above equations

$$\begin{aligned} \text{Longitudinal mass} &= \frac{m}{\left(1-\frac{V^2}{c^2}\right)^{\frac{3}{2}}} \\ \text{Transverse mass} &= \frac{m}{1-\frac{V^2}{c^2}} \end{aligned}$$

With a different definition of force and acceleration we should naturally obtain other values for the masses. This shows us that in comparing different theories of the motion of the electron we must proceed very cautiously. (Einstein 1905, pp. 62–63)

Einstein's caution is something that is worth taking seriously—not only from the perspective of obtaining values for mass but also from the perspective of our present concern about the principle of relativity. For notice that when we compare the descriptions of a physical system, say a charged particle moving in an electromagnetic field, expressed in terms of different frames of reference, what we actually do is exactly "comparing different theories of the motion of the electron". Indeed, on any account of the relativity principle seen so far, the principle requires the comparison of two different narratives, an "unprimed" and a "primed" one, about the same phenomena, judging some of their respective equations as "being of the same form". Such a comparison, however, only makes sense if the equations to be compared are written down in terms of the same variables, that is, in terms of physical quantities having the same meaning. It would be absurd to say, for example, that equations (2.38)–(2.40) and (2.41)–(2.43) "have the same form" if in the second set of equations E'_x, E'_y, E'_z denoted the components of the magnetic field strength rather then the electric one. Only when we have the *same* physical variables at the corresponding places in the equations to be compared can we say that they are "of the same form", in line with what the relativity principle requires.

It is also clear that even one and the same equation can be expressed in various forms in terms of different variables. For example, one can write down equations (2.44)–(2.46) in the form of

$$m_{\|}\frac{d^2x}{dt^2} = \epsilon E_x \tag{2.59}$$

$$m_{\|}\frac{d^2y}{dt^2} = \epsilon\frac{E_y - VB_z}{1-\frac{V^2}{c^2}} \tag{2.60}$$

$$m_{\|}\frac{d^2z}{dt^2} = \epsilon\frac{E_z + VB_y}{1-\frac{V^2}{c^2}} \tag{2.61}$$

where we introduced $m_{\parallel}$ for the quantity Einstein calls longitudinal mass. Now in a syntactic/typographic sense equations (2.44)–(2.46) and (2.59)–(2.61) have apparently *different* forms —for example they have different powers of $(1 - V^2/c^2)^{-\frac{1}{2}}$ in them. It is clear, however, that only when the meanings of the terms are taken into account does it make sense to compare the two sets of equations. Only when the *same* variable for mass is plugged into both equations, only then can we ascertain that the two equations are in fact identical.

The same point is emphasized by Grøn and Vøyenli (1999, p. 1731) in the context of the generalized principle of relativity:

> All quantities appearing in a covariant equation, must be defined in the same way in every coordinate system, and interpreted physically without reference to any preferred system. [...] A law fulfilling the restricted covariance principle, has the same mathematical form in every coordinate system, and it expresses a physical law that may be formulated by the same words (without any change of meaning) in every reference frame [...]

How are then the meanings of physical variables identified? And how can the corresponding unprimed and primed pairs of variables having the same physical meaning be matched together? How do we know that it is m and m', and $m_{\parallel}$ and $m'_{\parallel}$ that correspond to each other and not the other way round? Einstein suggests, rather offhandedly, that the meaning of mass is determined by the equation "mass × acceleration = force"; in the sense that mass is meant to be *defined* as the quantity that makes the equation of motion to take the form "mass × acceleration = force". Now suppose that the same definition is applied in another, "primed" frame, saying mass' is the quantity that makes the equation of motion in the primed frame to take the form "mass' × acceleration' = force"'. Does this mean that the equation of motion will take the same form in the two frames *merely due to the definition of mass*?

13. A similar but more explicit reasoning appears in the 1905 paper when Einstein considers the relativistic behavior of light rays. He writes:

> Since $A^2/8\pi$ equals the energy of light per unit of volume, we have to regard $A'^2/8\pi$, by the principle of relativity, as the energy of light in the moving system. (Einstein 1905, p. 57)

If we are to *regard* the energy of light per unit of volume as being the same expression of the amplitude A of the light wave in the two reference frames in question, then these *definitions* will certainly be of the same form in the two frames. But what does this *analytic truth* have to do with what the relativity principle is meant to *state*?

14. Isn't this thoroughly circular? The principle of relativity requires that certain equations, expressed in terms of quantities associated with different reference frames, must have the same form. The concept of "having the same form" requires a previous identification of quantities associated with different frames on the basis of their identical meaning. Now this identification is proposed to be constituted by the fact that the corresponding quantities play identical typographic/algebraic/structural roles in *equations of the same form*. Such an idea would render the relativity principle either meaningless or tautological, it is not even clear which one.

It seems obvious that if the principle of relativity is a statement with physical content, then at least some of the physical quantities featuring in our physical narrative must be capable of being given meaning independently of the equations they enter and which the principle concerns. Such a way of meaning assignment is indicated by Einstein when he parenthetically remarks, in the quoted passage in Paragraph **12**, that force can be measured by a spring balance. It is the operational definition of physical quantities that can provide this necessary "plugin" for the relativity principle.

15. The obvious solution is, therefore, that we identify those physical quantities, measured by different observers in different frames of reference, which have identical operational definitions. It is however far from obvious how these identical operational definitions are actually understood. For the empirical/operational definitions require *etalon* measuring equipments. But how do the observers in different reference frames share these *etalon* measuring equipments? Do they all base their definitions on the same *etalon* measuring equipments? They must do something like that, otherwise any comparison between their observations would be meaningless. But is the principle of relativity really understood in this way? Is it true that the laws of physics in K and K', which ought to take the same form, are expressed in terms of physical quantities defined/measured with the same standard measuring equipments? Not exactly. Consider how Einstein describes a simple application of the relativity principle in the kinematic part of the 1905 paper:

> Let there be given a stationary rigid rod; and let its length be l as measured by a measuring-rod which is also stationary. We now imagine the axis of the rod lying along the axis of x of the stationary system of co-

> ordinates, and that a uniform motion of parallel translation with velocity v along the axis of x in the direction of increasing x is then imparted to the rod. We now inquire as to the length of the moving rod, and imagine its length to be ascertained by the following two operations:
>
> (a) The observer moves together with the given measuring-rod and the rod to be measured, and measures the length of the rod directly by superposing the measuring-rod, in just the same way as if all three were at rest.
>
> (b) By means of stationary clocks set up in the stationary system and synchronizing in accordance with [the light-signal synchronization], the observer ascertains at what points of the stationary system the two ends of the rod to be measured are located at a definite time. The distance between these two points, measured by the measuring-rod already employed, which in this case is at rest, is also a length which may be designated "the length of the rod."
>
> In accordance with the principle of relativity the length to be discovered by the operation (a)—we will call it "the length of the rod in the moving system"—must be equal to the length l of the stationary rod.
>
> The length to be discovered by the operation (b) we will call "the length of the (moving) rod in the stationary system." This we shall determine on the basis of our two principles, and we shall find that it differs from l. (Einstein 1905, pp. 41–42)

That is to say, if the standard measuring equipment by means of which the observer in K defines a physical quantity ξ is at rest in K and, therefore, moving in K', then the observer in K' does not define the corresponding ξ' as the physical quantity obtainable by means of the original standard equipment—being at rest in K and moving in K'—but rather as the physical quantity obtainable by means of the standard equipment *in another state of motion*; namely, at rest relative to K' and in motion relative to K. Consequently the measurement operations and the measurement outcomes in K' are not the same physical phenomena as their counterparts in K. Therefore, the physical meanings of the variables that we identify when comparing equations "of the same form", are, in fact, *different*.

16. "The laws of physics have the same form in all inertial frames of reference"—this is how the principle of relativity reads in its most widespread formulation. This statement is usually regarded as a simple and clear expression of what we take to be the essential ingredient of Einstein's special relativity. However, once we slow done and try to understand more carefully what the principle actually says, we face a more confusing picture. In the 1905 paper, Einstein himself applies the relativity principle in various different ways, none of them being completely unproblematic in its own terms, and, more importantly, none of them being equivalent with one another, some of them being even contradictory together.

In trying to understand the precise meaning of the principle one encounters several obvious questions.

- What is counted as a "law of physics" here—for example, the Maxwell equations, constituting the totality of universally valid statements of a theory; or any part of it, like Faraday's law; or a Coulomb solution, describing a concrete physical situation; or a formula describing a particular aspect of such a situation, like the value of current in a wire loop?

- In what sense can a law of physics be "in" an inertial frame of reference—should the physical system which the law concerns co-move with the frame of reference in question; or should the law simply be a formula written down in terms of physical variables that are associated with the frame?

- How do we associate physical quantities with frames of reference?

- Of course, it is the *same* laws of physics which must take the same form in all inertial frames. Again, it would be absurd to require that, say, the second law of thermodynamics in K must have the same form as Newton's force law in K'. But, what are the same laws of physics in different inertial frames? How to identify a physical law, and how to identify its counterpart in another reference frame? For example, should the counterparts express the "same thing", that is, concern the *same* phenomenon, exhibited by the *same* system, like the behavior of the magnet and conductor in a *given* state of motion? Or should the relativity principle rather be seen as a comparison of laws describing *different* phenomena, like the comparison of the law of motion of the charge *at rest versus in motion*?

- What does it take to be of the "same form"?—one and the same physical law can take completely different forms/shapes in some algebraic/typographic sense when expressed in terms of different physical variables; and even when expressed in terms of a fixed set of variables, it can have various logically equivalent formulations that do not have the "same form".

- How do we identify a physical variable, and how do we identify its counterpart in another reference frame? If they are identified by means of their operational definitions, how are the etalons and the measuring devices shared between the different reference frames?

This chapter is devoted to clarify these and similar questions.

2.2 The Electromagnetic Field of a Static versus Uniformly Moving Charge

17. Of the various different, sometimes even contradictory, understandings of the principle of relativity we have been considering, what are the ones then that we can take seriously? In light of the severe ambiguities the comparison of the various formulations and applications reveals, what exactly are the ingredients we ought to regard as part of the statement of the relativity principle? To make the first step in clarifying what the principle actually asserts, I would like to consider one further example—one of the few with respect to which there seems to be an overall consensus in the literature on its being a clear case of the relativity principle. This is the well-known textbook example of the electromagnetic field of a static versus uniformly moving charged particle.

The static field of a charge q being at *rest* at point (x_0, y_0, z_0) in an inertial frame of reference K is the following (Fig. 2.1a):

$$\begin{aligned}
E_x(x,y,z,t) &= \frac{q\,(x-x_0)}{\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\
E_y(x,y,z,t) &= \frac{q\,(y-y_0)}{\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\
E_z(x,y,z,t) &= \frac{q\,(z-z_0)}{\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\
B_x(x,y,z,t) &= 0 \\
B_y(x,y,z,t) &= 0 \\
B_z(x,y,z,t) &= 0
\end{aligned} \tag{2.62}$$

The stationary field of a charge q *moving* at constant velocity $\mathbf{V} = (V, 0, 0)$ relative to K can be obtained (Jackson 1999, pp. 661–665) by solving the equations

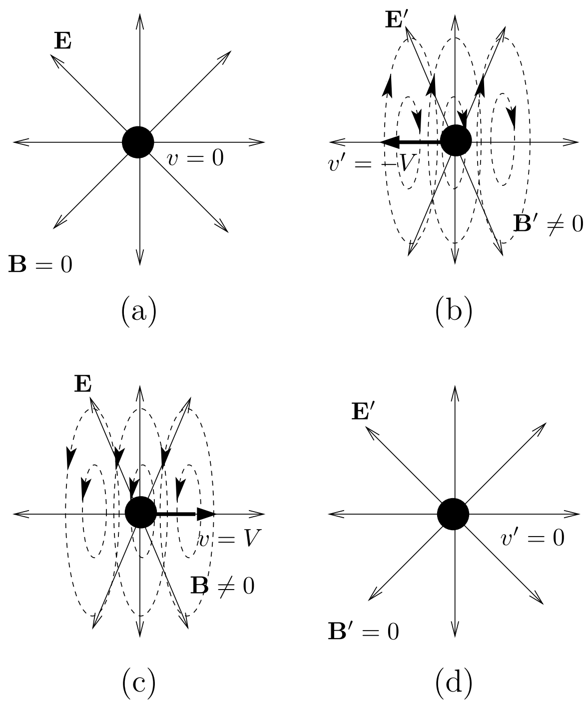

Figure 2.1: The descriptions of the same phenomenon in different inertial frames are different: (a) and (b). In contrast, different phenomena, (a) and (c), have descriptions of the same form in the two different (co-moving) inertial frames: (a) and (d)

of electrodynamics (in K) with the time-depending source (Fig. 2.1c):

$$\begin{aligned}
E_x(x,y,z,t) &= \frac{qX_0}{\left(X_0^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}} \\
E_y(x,y,z,t) &= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \frac{q\,(y-y_0)}{\left(X_0^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}} \\
E_z(x,y,z,t) &= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \frac{q\,(z-z_0)}{\left(X_0^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}} \\
B_x(x,y,z,t) &= 0 \\
B_y(x,y,z,t) &= -\frac{V}{c^2} E_z(x,y,z,t) \\
B_z(x,y,z,t) &= \frac{V}{c^2} E_y(x,y,z,t)
\end{aligned} \tag{2.63}$$

where (x_0, y_0, z_0) is the initial position of the particle at $t = 0$, $X_0 = \frac{x-(x_0+Vt)}{\sqrt{1-V^2/c^2}}$.

Now, we form the same expressions as (2.62) but in the *primed* variables of reference frame K' co-moving with the charge (Fig. 2.1d):

$$\begin{aligned}
E'_x(x',y',z',t') &= \frac{q'\,(x'-x_0)}{\left((x'-x_0)^2+(y'-y_0)^2+(z'-z_0)^2\right)^{3/2}}\\
E'_y(x',y',z',t') &= \frac{q'\,(y'-y_0)}{\left((x'-x_0)^2+(y'-y_0)^2+(z'-z_0)^2\right)^{3/2}}\\
E'_z(x',y',z',t') &= \frac{q'\,(z'-z_0)}{\left((x'-x_0)^2+(y'-y_0)^2+(z'-z_0)^2\right)^{3/2}}\\
B'_x(x',y',z',t') &= 0\\
B'_y(x',y',z',t') &= 0\\
B'_z(x',y',z',t') &= 0
\end{aligned} \tag{2.64}$$

By means of the Lorentz transformation rules of the space-time coordinates, the field strengths and the electric charge, one can express (2.64) in terms of the original variables of K (Tolman 1917, p. 177) (Fig. 2.1c):

$$\begin{aligned}
E_x(x,y,z,t) &= \frac{qX_0}{\left(X_0^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}}\\
E_y(x,y,z,t) &= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\frac{q\,(y-y_0)}{\left(X_0^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}}\\
E_z(x,y,z,t) &= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\frac{q\,(z-z_0)}{\left(X_0^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}}\\
B_x(x,y,z,t) &= 0\\
B_y(x,y,z,t) &= -\frac{V}{c^2}E_z(x,y,z,t)\\
B_z(x,y,z,t) &= \frac{V}{c^2}E_y(x,y,z,t)
\end{aligned} \tag{2.65}$$

We find that the result is the same as (2.63) describing the field of the moving charge. This agreement is what we count as the satisfaction of the principle of relativity in this particular case. One can express this fact by saying that the field of the charge at rest in K', described in terms of K' ((2.64) and Fig. 2.1d), is of the same form as the field of the charge when it is at rest in K, described in terms of K ((2.62) and Fig. 2.1a).

Reversely, *assuming* that the particle + electromagnetic field system satisfies the relativity principle, one can, by this method, *derive* the stationary field of a uniformly moving point charge (2.63) from the static field (2.62). As a prototypical application of the principle of relativity, this is the derivation we find in almost every textbook on relativity theory and electrodynamics (e.g. Tolman 1917, pp. 176–178; Møller 1955, pp. 151–154; Pauli 1958, pp. 91–92; Feynman, Leighton and Sands 1964, pp. 25-9–25-10; Landau and Lifshitz 1980, pp. 98–99; Griffiths 1999, p. 533; Sardesai 2004, pp. 158–161; Rindler 2006, pp. 148–149).

18. On the basis of the above example one can make the following observations regarding the clarificatory questions posed in Paragraph **16**.

- The above application of the relativity principle is about the agreement of two different methods for obtaining the description of the charge + field system *in a concrete physical situation*. Notice that this *particular* application of the principle would remain valid even if the *general equations of electrodynamics*, from which the description of the concrete situation in question derives as a solution, looked different, or even if they did not happen to be Lorentz covariant—as long as the particular behaviors of the system considered still respected these laws. In fact, the example can be retold without even referring to the general laws of electrodynamics—just take the form of the electromagnetic field of the moving charge, as well as that of the static charge, as empirical facts of the world. Consequently, the very statement of the relativity principle cannot invoke the notion of a general law; the *primary* subjects of the relativity principle cannot be the general equations of physics. Rather, the principle must be construed as a statement about the descriptions of concrete physical situations/phenomena—the charge and its field in specific states of motion; in other words, a statement about *particular solutions* of the general laws of physics. If the relativity principle has anything to say about the properties of general laws, this statement must be a consequence of what the principle says about the solutions of these general equations that describe concrete physical situations.

- Thus, a “law of physics” in the formula of the principle must be understood as a description of a concrete physical phenomenon. If so, then one might think it is the description of the *same* phenomenon which must have the same form in different inertial frames of reference. However, as it is clear from the above example, this is obviously not the case. Consider the electromagnetic field of the charge at rest in K. This phenomenon is described in K as it is depicted in Fig. 2.1a. Now, the description of the *same* phenomenon in K' is completely different, Fig. 2.1b—just take the Lorentz transformation of the situation in Fig. 2.1a. Thus, the opposite must be true: the relativity principle is about *different* physical phenomena; *different* phenomena must have descriptions of the same form in the different inertial frames of reference. In our example, 'the static field of the rest charge' is *one* phenomenon (Fig. 2.1a) and 'the time-dependent stationary field of the same charge in motion with velocity $\mathbf{V} = (V, 0, 0)$' is the *other* (Fig. 2.1c). The subjects of the relativity principle, the pair of phenomena that must have the same descriptions in different frames, are the same phenomenon *except that in one case the physical system exhibiting the phenomenon co-moves, as a whole, with a frame K, whereas in the other case it co-moves with another frame K'.*

- It is clear that a law of physics "*in*" a reference frame K, that is, a description of a physical phenomenon "*in*" K, is meant to be the description as it is ascertained by an observer living in reference frame K; less anthropomorphically, as it appears in the results of measurements such that the measuring equipments are *at rest relative to* K. In other words, a description in K is a formula expressed in terms of physical quantities that are measured by means of measuring devices at rest in K. So, when we say that the electromagnetic field of the moving charge is of form (2.63) *in* K and of form (2.64) *in* K', what we mean is that these descriptions obtain as a result of measuring the values of the electric and magnetic field strengths, the charge of the particle, etc., with measuring equipments being at rest relative to K and K', respectively. This "operational" understanding of the association of physical descriptions and physical quantities with frames of reference is suggested by Einstein when he remarks, in the quoted passage in Paragraph **12**, that the components $q'E'_x, q'E'_y, q'E'_z$ of the ponderomotive force acting upon a charged particle can be measured by a spring balance *at rest in the "moving" frame K'.*[6]

- The typographic sense in which we judge formulas (2.62) and (2.64) as being "of the same form", in the unprimed and primed variables respectively, simply means that (2.64) can be obtained by putting a prime on every unprimed quantity in (2.62). As discussed in Paragraph **12** and **14**, however, this notion requires a previous identification of physical quantities in different frames of reference. Now, as a physical quantity "*in*" a frame of reference is meant to be the quantity measured by means of measuring equipments at rest relative to the frame, there is a natural way to construe such an identification. If $\mathbf{E}$, the

[6]Notice that all this implies: only physical laws/descriptions formulated in terms of *measurable* quantities can be subjected to the relativity principle. In light of this fact some standard cases where the principle is applied for laws/descriptions of non-empirical nature seem problematic. The simplest example is the case of the electromagnetic field of the moving charge reformulated in terms of the electromagnetic potentials, as we find it for example in Feynman, Leighton and Sands 1964 (pp. 25-9–25-10).

electric field strengths in frame K, is ascertained by measuring the acceleration of the unit charge *at rest in* K, then its counterpart $\mathbf{E}'$ in frame K' is the quantity ascertained by measuring the acceleration of the unit charge *at rest in* K'—such that everything else in these measurements, including the clocks and rods with which the accelerations are measured, is at rest in K and K' respectively. More generally, in line with what has been said in Paragraph **15**, we identify those physical quantities in frames of reference K and K' which are measured by means of the same operational procedures, with the same measuring equipments, *except that in one case everything co-moves with frame* K*, whereas in the other case everything co-moves with frame* K'

19. Putting these considerations together, as a first step towards the precise formulation, we give a preliminary formulation of the relativity principle (following Szabó 2004, p. 481):

> *The description of a phenomenon exhibited by a physical system co-moving as a whole with an inertial frame* K*, expressed in terms of the results of measurements obtainable by means of measuring equipments co-moving with* K*, takes the same form as the description of the same phenomenon exhibited by the same physical system, except that the system is co-moving with another inertial frame* K'*, expressed in terms of the measurements with the same equipments when they are co-moving with* K'*.*

2.3 Conceptual Components of the Relativity Principle

20. Let us now unpack this verbal formulation in a more mathematical way. The formal reconstruction will make explicit all the conceptual "plugins" that are necessary in order for statement of the relativity principle to be precisely formulated.

Let K and K' be two arbitrary inertial frames of reference; and let $\mathbf{V}$ be the velocity of K' relative to K.[7] An essential requisite of the relativity principle is an association of physical quantities with frames of reference. More precisely:

Measurement outcomes in K Denote by Σ the set of all possible measurement operations with certain measuring devices being *at rest*

[7] That is, $\mathbf{V} \in \sigma_s$ for the appropriate $s \in \Sigma$ corresponding to a velocity measurement, as defined below. More precisely, the components of $\mathbf{V}$ may be conceived as the corresponding real valued quantities ξ_i to be introduced in Paragraph **25**.

in inertial frame K. Let σ_s denote the set of the possible outcomes of measurement $s \in \Sigma$. We assume that $\sigma_{s_1} \cap \sigma_{s_2} = \emptyset$ for all $s_1 \neq s_2$. Let E denote the union of all possible outcomes of all possible measurements: $E \overset{def}{=} \bigcup_{s\in\Sigma} \sigma_s$.

Measurement outcomes in K' Similarly, denote by Σ' the set of all possible measurement operations with certain measuring devices being *at rest* in inertial frame K'. Let $\sigma'_{s'}$ denote the set of the possible outcomes of measurement $s' \in \Sigma'$. We assume that $\sigma'_{s'_1} \cap \sigma'_{s'_2} = \emptyset$ for all $s'_1 \neq s'_2$. Finally, $E' \overset{def}{=} \bigcup_{s'\in\Sigma'} \sigma'_{s'}$.

The relativity principle requires an identification between the physical quantities in different inertial frames; we need to be able to say which measurement operation and measurement outcome in K correspond to which measurement operation and measurement outcome in K'. In other words:

The counterparts We need to have a pair of one-to-one maps

$$P_1 : E \to E'$$
$$P_2 : \Sigma \to \Sigma'$$

such that for all $s \in \Sigma$,

$$P_1(\sigma_s) = \sigma'_{P_2(s)} \tag{2.66}$$

21. We have to emphasize that P_1 and P_2 are not simply arbitrary one-to-one relations between the measurement operations and measurement outcomes somehow assigned to inertial frames K and K', satisfying (2.66), but they must express the following physically meaningful correspondence: $P_2(s)$ must be the same measurement operation with the same measuring equipment as measurement s, except that everything is in a collective motion with velocity $\mathbf{V}$ relative to K. Similarly, the measurement outcome $P_1(\omega_s)$ must be the same physical phenomenon as the measurement outcome ω_s, except that everything is in a collective motion with velocity $\mathbf{V}$ relative to K. For example, if measurement outcome ω_s consists in that the pointer of a measuring device at rest relative to K is in a certain position, then the measurement outcome $P_1(\omega_s)$ must be the phenomenon consisting in that the pointer of the same measuring device is in the same position, except that everything—the device, the pointer, the scale—is in a collective motion with velocity $\mathbf{V}$ relative to K.

22. For the sake of simplicity, in what follows we restrict the discussion for a finite number of measurements $s_1, s_2, \ldots s_n \in \Sigma$. Let Ω denote the set of the possible outcome combinations:

$$\Omega \stackrel{def}{=} \times_{i=1}^{n} \sigma_{s_i}$$

The counterparts are $P_2(s_1), P_2(s_2), \ldots P_2(s_n) \in \Sigma'$ and

$$\Omega' \stackrel{def}{=} \times_{i=1}^{n} \sigma'_{P_2(s_i)}$$

> **Putting primes** Maps P_1 and P_2 determine the following bijection between Ω and Ω':
>
> $$\begin{aligned} P: \quad & \Omega \to \Omega' \\ & (\omega_1, \omega_2, \ldots \omega_n) \mapsto (P_1(\omega_1), P_1(\omega_2), \ldots P_1(\omega_n)) \end{aligned} \tag{2.67}$$

23. In the above sense, the points of Ω and the points of Ω' range over all possible measurement outcome combinations $(\omega_1, \omega_2, \ldots \omega_n)$ and $(\omega'_1, \omega'_2, \ldots \omega'_n)$. It might be the case however that some combinations are impossible, in the sense that they never come to existence in the physical world.

> **Admissible values** Let us denote by $R \subseteq \Omega$ and $R' \subseteq \Omega'$ the physically admissible parts of Ω and Ω'.

Note that $P(R)$ is not necessarily identical with R'.[8]

24. For the formulation of the principle of relativity we need to introduce the concept of what we usually call the "transformation" of physical quantities.

> **Transformation law** It is conceived as a bijection between Ω and Ω', more precisely, between R and R':
>
> $$\Lambda : \Omega \supseteq R \to R' \subseteq \Omega' \tag{2.68}$$
>
> determined by the contingent fact—if there is such a fact—that whenever the measurements $s_1, s_2, \ldots s_n$ have outcomes $(\omega_1, \omega_2, \ldots \omega_n) \in R$ then, in the same physical constellation, the outcomes of the measurements $P_2(s_1), P_2(s_2), \ldots P_2(s_n)$ are $\Lambda(\omega_1, \omega_2, \ldots \omega_n) \in R'$, and vice versa.

For an arbitrary set of measurements $s_1, s_2, \ldots s_n$ nothing guarantees that such a bijection exists. Because the outcomes of measurements $s_1, s_2, \ldots s_n$, generally,

[8] One can show however that $P(R) = R'$ if the relativity principle, that is (2.74), holds.

do not specify a physical constellation in which the outcomes of measurements $P_2(s_1), P_2(s_2), \ldots P_2(s_n)$ are uniquely determined. (For a simple example, see Paragraph **26** (b) vs. (c).) In what follows we assume that Λ exists; if there were no such a bijection, the relativity principle could not be stated.

25. To bring our formalism closer to the ordinary language of physics, without serious loss of generality, we make the following assumption:

> **Numeric values** We assume that the measurement outcomes can be labeled by real numbers, by means of two coordinate maps $\phi : \Omega \to \mathbb{R}^n$ and $\phi' : \Omega' \to \mathbb{R}^n$; and, for the sake of convenience, we also assume that
>
> $$\phi(\omega) = \phi'(P(\omega)) \tag{2.69}$$
>
> for all $\omega \in \Omega$. Let us denote the coordinates by $(\xi_1, \xi_2, \ldots \xi_n) = \phi(\omega)$ and $(\xi'_1, \xi'_2, \ldots \xi'_n) = \phi'(\omega')$. We will refer to them as real valued *physical quantities* measured by the measurements $s_1, s_2, \ldots s_n$ and $P_2(s_1), P_2(s_2), \ldots P_2(s_n)$, respectively.

In spite of (2.69), it must be emphasized that $\xi_1, \xi_2, \ldots \xi_n$ and $\xi'_1, \xi'_2, \ldots \xi'_n$ are a priori *different* real valued physical quantities, due to the fact that the operations by which the quantities are defined are performed under *different* physical conditions. The same numeric values, say, $(5, 12, \ldots 61) \in \mathbb{R}^n$ generally correspond to different physical constellations when $\xi_1 = 5, \xi_2 = 12, \ldots \xi_n = 61$ versus $\xi'_1 = 5, \xi'_2 = 12, \ldots \xi'_n = 61$.

26. It is worthwhile to consider several examples.

(a) Let (ξ_1, ξ_2) be (p, T), the pressure and the temperature of a given (equilibrium) gas; and let (ξ'_1, ξ'_2) be (p', T'), the pressure and the temperature of the same gas, measured by the moving observer in K'. In this case, there exists a one-to-one Λ (Tolman 1949, pp. 158–159):[9]

$$p' = p \tag{2.70}$$

$$T' = T\sqrt{1 - \frac{V^2}{c^2}} \tag{2.71}$$

A point $\omega \in \Omega$ of coordinates, say, $p = 101325$ and $T = 300$ (in units Pa and $^\circ K$) represents the physical constellation in which the gas in question has pressure of $101325\, Pa$ and temperature of $300\,^\circ K$. Due to (2.71), this physical

[9] There is a debate over the proper transformation rules (Georgieu 1969; Sewell 2008).

constellation is different from the one represented by $P(\omega) \in \Omega'$ of coordinates $p' = 101325$ and $T' = 300$; but it is identical to the one represented by $\Lambda(\omega) \in \Omega'$ of coordinates $p' = 101325$ and $T' = 300\sqrt{1 - V^2/c^2}$. Here we can see the difference between bijections Λ and P.

(b) Let $(\xi_1, \xi_2, \ldots \xi_{10})$ be $(x, y, z, t, E_x, E_y, E_z, r_x, r_y, r_z)$, the time, the space coordinates where the electric field strength is taken, the three components of the field strength, and the space coordinates of a particle. And let $(\xi'_1, \xi'_2, \ldots \xi'_{10})$ be $\left(x', y', z', t', E'_x, E'_y, E'_z, r'_x, r'_y, r'_z\right)$, the similar quantities obtainable by means of measuring equipments co-moving with K'. In this case, there is no suitable one-to-one Λ, as the electric field strength in K does not determine the electric field strength in K', and vice versa.

(c) Let $(\xi_1, \xi_2, \ldots \xi_{13})$ be $(x, y, z, t, E_x, E_y, E_z, B_x, B_y, B_z, r_x, r_y, r_z)$ and let $(\xi'_1, \xi'_2, \ldots \xi'_{13})$ be $\left(x', y', z', t', E'_x, E'_y, E'_z, B'_x, B'_y, B'_z, r'_x, r'_y, r'_z\right)$, where B_x, B_y, B_z and B'_x, B'_y, B'_z are the magnetic field strengths in K and K'. In this case, in contrast to (b), the well known Lorentz transformations of the spatio-temporal coordinates (4.39)–(4.42) and the electric and magnetic field strengths (2.28)–(2.33) constitute a proper one-to-one Λ.

27. Next we turn to the general formulation of the concept of *description of a particular phenomenon* exhibited by a physical system, in terms of physical quantities $\xi_1, \xi_2, \ldots \xi_n$ in K. We are probably not far from the truth if we stipulate the following:

> **Description of a phenomenon** Such a description, in its most abstract sense, is a *relation* between physical quantities $\xi_1, \xi_2, \ldots \xi_n$; in other words, it can be given as a subset $F \subset R$.

Consider the above example (a) in Paragraph **26**. An isochoric process of the gas can be described by the subset F that is, in ϕ-coordinates, determined by the following single equation:

$$F = \left\{p = \kappa T\right\} \tag{2.72}$$

with a certain constant κ.

To give another example, consider the case (b). The relation F given by equations

$$F = \left\{ \begin{array}{rcl} E_x(x,y,z,t) & = & E_0 \\ E_y(x,y,z,t) & = & 0 \\ E_z(x,y,z,t) & = & 0 \\ r_x(t) & = & r_0 + v_0 t \\ r_y(t) & = & 0 \\ r_z(t) & = & 0 \end{array} \right\} \tag{2.73}$$

with some specific values of E_0, r_0, v_0 describes a neutral particle moving with constant velocity in a static homogeneous electric field.

28. Of course, one may not assume that an arbitrary relation $F \subset R$ has physical meaning; in the sense that it corresponds to an actual behavior exhibited by the system in question.

> **Physical equations** Let $\mathcal{E} \subset 2^R$ be the set of those $F \subset R$ which describe a particular behavior of the system. We shall call $\mathcal{E}$ the *set of equations* describing the physical system in question.

The term is entirely justified. In practical calculations, two systems of equations are regarded as equivalent if and only if they have the same solutions. Therefore, a system of equations can be identified with the set of its solutions. In general, the equations can be algebraic equations, ordinary and partial integro-differential equations, linear and nonlinear, whatever. So, in its most abstract sense, a system of equations formulated in terms of physical quantities $\xi_1, \xi_2, \ldots \xi_n$ is a set of subsets of R.

29. Now, consider the following subsets of Ω', determined by an $F \in \mathcal{E}$:

> **Primed solution** $P(F) \subset \Omega'$: the "primed F", that is a relation "of exactly the same form as F, but in the primed variables $\xi'_1, \xi'_2, \ldots \xi'_n$".
>
> **Same solution expressed in primed variables** $\Lambda(F) \subset R'$: which is the same description of the same physical situation as F, but *expressed* in the primed variables.

The quotation marks in the first stipulation are important. Since one and the same $F \subset \Omega$ can be given in many different "forms", by means of different numbers of different equations, functions, relations, of different types. That is why we formalized the concept of a description of a phenomenon as an abstract relation between quantities $\xi_1, \xi_2, \ldots \xi_n$, given in the form of a subset of Ω. Similarly, subset $P(F)$ is an abstract relation between $\xi'_1, \xi'_2, \ldots \xi'_n$, which can be thought of in many different equivalent "forms". So, whether F and $P(F)$ are "of the same form" may or may not be manifestly apparent (cf. Friedman 1983, p. 150). Also note that relation $P(F)$ does not necessarily describe a true physical situation, since it can be not realized in nature.

30. In order to formulate the principle of relativity we need one more concept. The principle is about the connection between two situations: one is in which the system, as a whole, is at rest relative to inertial frame K, the other is in which the system shows the similar behavior, but being in a collective motion relative to K, co-moving with K'. In other words:

> **The same but in different state of motion** We assume the existence of a map $M : \mathcal{E} \rightarrow \mathcal{E}$, assigning to each $F \in \mathcal{E}$, stipulated to describe a phenomenon exhibited by a system co-moving with inertial frame K, another relation $M(F) \in \mathcal{E}$, that describes the same physical system exhibiting the same phenomenon as the one described by F, except that the system is *in motion* with velocity $\mathbf{V}$ relative to K, that is, co-moving with inertial frame K'.

2.4 The Formal Statement of the Relativity Principle

31. Now, applying all these concepts (Fig. 2.2), what the relativity principle states is the following:

$$\Lambda\left(M(F)\right) = P(F) \quad \text{for all } F \in \mathcal{E} \tag{2.74}$$

or equivalently,

$$P(F) \subset R' \text{ and } M(F) = \Lambda^{-1}\left(P(F)\right) \quad \text{for all } F \in \mathcal{E} \tag{2.75}$$

32. Notice that, for a given fixed F, everything on the right hand side of the equation in (2.75), P and Λ, are determined *only by* the physical behaviors of *the measuring equipments* when they are in various states of motion. In contrast, the meaning of the left hand side, $M(F)$, depends on the physical behavior of *the object physical system* described by F and $M(F)$, when it is in various states of motion.

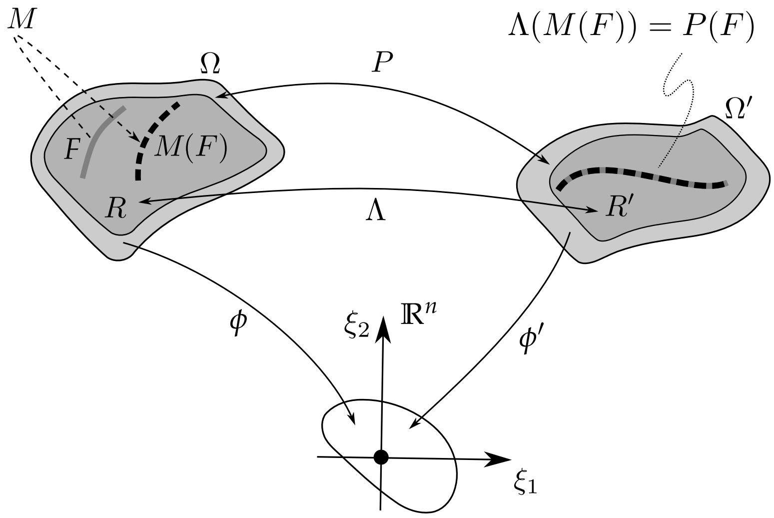

Figure 2.2: The relativity principle

That is to say, the two sides of the equation reflect the behaviors of *different parts* of the physical reality; and the relativity principle expresses a law-like regularity between the behaviors of these different parts.

33. To illustrate these concepts, let us repeat the case of the electromagnetic field of the static versus uniformly moving charge (Paragraph **17**). Let $(\xi_1, \xi_2, \ldots \xi_{14})$ be $(x, y, z, t, E_x, E_y, E_z, B_x, B_y, B_z, r_x, r_y, r_z, q)$, that is the space and time coordinates where the field strengths are taken, the components of the field strengths, and the space coordinates and charge of the particle—everything measured in K. Let $(\xi'_1, \xi'_2, \ldots \xi'_{14})$ be $\left(x', y', z', t', E'_x, E'_y, E'_z, B'_x, B'_y, B'_z, r'_x, r'_y, r'_z, q'\right)$, the similar quantities measured in K'. We start from the situation when the charge is at *rest* at point (x_0, y_0, z_0) in K (Fig. 2.3a). The corresponding F is, in ϕ-coordinates, determined by the formulas describing the static charge and its Coulomb field:[10]

[10]In what follows we will no longer adhere to the continuous indication of the difference between an $F \subset \Omega$, the corresponding $\phi(F) \subset \mathbb{R}^n$, and the actual numeric equations/formulas determining such a subset—where concrete physical equations/formulas appear, we automatically take them to identify the corresponding relations on the appropriate Ω space. Subsets $\mathcal{E} \subset 2^R$ as well as transformations P, Λ and M will be treated in a similar manner.

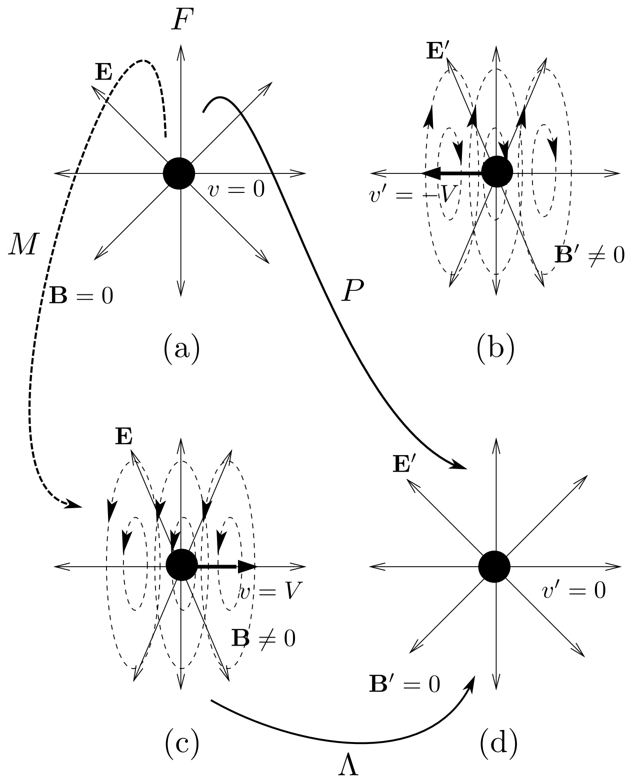

Figure 2.3: The relativity principle is true for the case of the electromagnetic field of the static versus uniformly moving charge. Let F denote the description of the static charge and its Coulomb field. Then: $\Lambda\,(M(F)) = P(F)$

$$F = \left\{ \begin{array}{l} E_x(x,y,z,t) = \dfrac{q\,(x-x_0)}{\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\ E_y(x,y,z,t) = \dfrac{q\,(y-y_0)}{\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\ E_z(x,y,z,t) = \dfrac{q\,(z-z_0)}{\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\ B_x(x,y,z,t) = 0 \\ B_y(x,y,z,t) = 0 \\ B_z(x,y,z,t) = 0 \\ \qquad r_x\,(t) = x_0 \\ \qquad r_y\,(t) = y_0 \\ \qquad r_z\,(t) = z_0 \end{array} \right\} \tag{2.76}$$

This is the solution of the basic equations of electrodynamics $\mathcal{E}$ that describes the charge + field system as it is, as a whole, at rest relative to K.

The equations describing the system when it is set in a collective *motion* with constant velocity $\mathbf{V} = (V, 0, 0)$ relative to K can also be obtained from the laws of electrodynamics in K (Fig. 2.3c):

$$M(F) = \left\{ \begin{array}{l} E_x(x,y,z,t) = \dfrac{qX_0}{\left(X_0^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\ E_y(x,y,z,t) = \dfrac{1}{\sqrt{1-\frac{V^2}{c^2}}} \dfrac{q\,(y-y_0)}{\left(X_0^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\ E_z(x,y,z,t) = \dfrac{1}{\sqrt{1-\frac{V^2}{c^2}}} \dfrac{q\,(z-z_0)}{\left(X_0^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\ B_x(x,y,z,t) = 0 \\ B_y(x,y,z,t) = -\dfrac{V}{c^2} E_z(x,y,z,t) \\ B_z(x,y,z,t) = \dfrac{V}{c^2} E_y(x,y,z,t) \\ \qquad r_x\,(t) = x_0 + Vt \\ \qquad r_y\,(t) = y_0 \\ \qquad r_z\,(t) = z_0 \end{array} \right\} \tag{2.77}$$

where, again, $X_0 = \frac{x-(x_0+Vt)}{\sqrt{1-V^2/c^2}}$.

Now, we form the same expressions as (2.76) but in the *primed* variables

$(\xi_1', \xi_2', \ldots \xi_{14}')$ of the co-moving reference frame K' (Fig. 2.3d):

$$P(F) = \left\{ \begin{aligned} E_x'(x', y', z', t') &= \frac{q'(x' - x_0)}{\left((x' - x_0)^2 + (y' - y_0)^2 + (z' - z_0)^2\right)^{3/2}} \\ E_y'(x', y', z', t') &= \frac{q'(y' - y_0)}{\left((x' - x_0)^2 + (y' - y_0)^2 + (z' - z_0)^2\right)^{3/2}} \\ E_z'(x', y', z', t') &= \frac{q'(z' - z_0)}{\left((x' - x_0)^2 + (y' - y_0)^2 + (z' - z_0)^2\right)^{3/2}} \\ B_x'(x', y', z', t') &= 0 \\ B_y'(x', y', z', t') &= 0 \\ B_z'(x', y', z', t') &= 0 \\ r_x'(t') &= x_0 \\ r_y'(t') &= y_0 \\ r_z'(t') &= z_0 \end{aligned} \right\} \tag{2.78}$$

By means of the Lorentz transformation rules of the space-time coordinates, the field strengths and the electric charge, Λ, one can express (2.78) in terms of the original variables of K (Fig. 2.3c):

$$\Lambda^{-1}(P(F)) = \left\{ \begin{aligned} E_x(x, y, z, t) &= \frac{qX_0}{\left(X_0^2 + (y - y_0)^2 + (z - z_0)^2\right)^{3/2}} \\ E_y(x, y, z, t) &= \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \frac{q(y - y_0)}{\left(X_0^2 + (y - y_0)^2 + (z - z_0)^2\right)^{3/2}} \\ E_z(x, y, z, t) &= \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \frac{q(z - z_0)}{\left(X_0^2 + (y - y_0)^2 + (z - z_0)^2\right)^{3/2}} \\ B_x(x, y, z, t) &= 0 \\ B_y(x, y, z, t) &= -\frac{V}{c^2} E_z(x, y, z, t) \\ B_z(x, y, z, t) &= \frac{V}{c^2} E_y(x, y, z, t) \\ r_x(t) &= x_0 + Vt \\ r_y(t) &= y_0 \\ r_z(t) &= z_0 \end{aligned} \right\} \tag{2.79}$$

We find that the result is indeed the same as (2.77) describing the field of the moving charge: $M(F) = \Lambda^{-1}(P(F))$. In other words, the field of the moving charge expressed in terms of K', given by $\Lambda(M(F))$, is "of the same form" as the field of

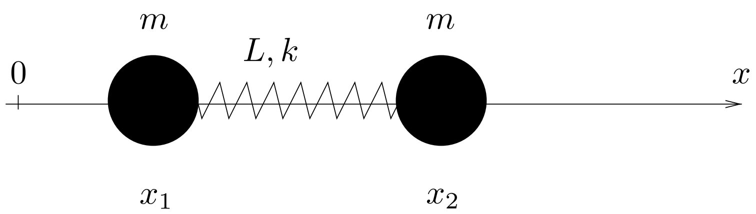

Figure 2.4: Two point masses of equal mass m are connected by a spring of equilibrium length L and of spring constant k

the static charge in terms of K, given by F; that is: $\Lambda\left(M(F)\right) = P(F)$ (Fig. 2.3). Thus, the principle of relativity is true in this particular case.

34. To give an another example (Szabó 2004, pp. 484–485), consider a non-relativistic system consisting of two point particles of equal mass connected by a spring (Fig. 2.4). Let $(\xi_1, \xi_2, \ldots \xi_6)$ be (t, x_1, x_2, m, L, k), the time coordinate, the space coordinates of the one-dimensional motions of the two particles, their identical mass, the equilibrium length of the spring and the spring constant in inertial frame K. Let $(\xi_1', \xi_2', \ldots \xi_6')$ be $(t', x_1', x_2', m', L', k')$, the similar quantities in reference frame K' moving with velocity V relative to K along the x-axis. The equations of motion describing the system in inertial frame K are

$$\mathcal{E} = \left\{ \begin{array}{c} m\dfrac{d^2x_1\left(t\right)}{dt^2} = k\left(x_2\left(t\right) - x_1\left(t\right) - L\right) \\ m\dfrac{d^2x_2\left(t\right)}{dt^2} = -k\left(x_2\left(t\right) - x_1\left(t\right) - L\right) \\ x_2\left(t\right) > x_1\left(t\right) \end{array} \right\} \tag{2.80}$$

Consider the following solution F of these equations, stipulated the describe the behavior of the system when it is, as a whole, at *rest* in K:

$$F = \left\{ \begin{array}{l} x_1\left(t\right) = -\dfrac{A}{2}\cos\left(\sqrt{\dfrac{2k}{m}}t\right) - \dfrac{L}{2} \\ x_2\left(t\right) = \dfrac{A}{2}\cos\left(\sqrt{\dfrac{2k}{m}}t\right) + \dfrac{L}{2} \end{array} \right\} \tag{2.81}$$

(2.81) corresponds to a symmetric oscillatory motion of the two particles with amplitude $A/2$, one around $-L/2$, the other one around $L/2$ $(A < L)$, with their center of mass sitting at the origin of the x-axis.

Now form the same expressions as (2.81) but in the *primed* variables $(\xi_1', \xi_2', \ldots \xi_6')$:

$$P\left(F\right)=\left\{\begin{aligned} x_1'\left(t'\right) &= -\frac{A}{2}\cos\left(\sqrt{\frac{2k'}{m'}}t'\right)-\frac{L'}{2}\\ x_2'\left(t'\right) &= \frac{A}{2}\cos\left(\sqrt{\frac{2k'}{m'}}t'\right)+\frac{L'}{2}\end{aligned}\right\} \tag{2.82}$$

One can express (2.82) in terms of the original variables of K by means of the Galilean transformations for x_1, x_2 and t, along with the non-relativistic transformation rules of m, L and k,

$$\Lambda=\left\{\begin{aligned} x_1' &= x_1 - Vt\\ x_2' &= x_2 - Vt\\ t' &= t\\ m' &= m\\ L' &= L\\ k' &= k\end{aligned}\right\} \tag{2.83}$$

and obtain:

$$\Lambda^{-1}\left(P(F)\right)=\left\{\begin{aligned} x_1\left(t\right) &= -\frac{A}{2}\cos\omega t-\frac{L}{2}+Vt\\ x_2\left(t\right) &= \frac{A}{2}\cos\omega t+\frac{L}{2}+Vt\end{aligned}\right\} \tag{2.84}$$

Apparently, the result is nothing but $M(F)$, that is the solution of $\mathcal{E}$ describing the same evolution of the system as (2.81), but when it is, as a whole, together with its center of mass, in an additional translatory *motion* of velocity V along the x-axis. Again, $M(F)=\Lambda^{-1}\left(P(F)\right)$, that is, the principle of relativity is satisfied.

2.5 Covariance

35. Now we have a strict mathematical formulation of the relativity principle for a physical system described by a system of equations $\mathcal{E}$. Remarkably, however, we still have not encountered the concept of "covariance" of equations $\mathcal{E}$. The reason is that the relativity principle and the covariance of equations $\mathcal{E}$ are not equivalent—in contrast to what is so often claimed in the literature. As Norton (1993, p. 796) writes:

> The lesson of Einsteins's 1905 paper was simple and clear. To construct a physical theory that satisfied the principle of relativity of inertial motion,

it was sufficient to ensure that it had a particular formal property: its laws must be Lorentz covariant. Lorentz covariance became synonymous with satisfaction of the principle of relativity of inertial motion and the whole theory itself, as Einstein (1940, p. 329) later declared:

> The content of the restricted relativity theory can accordingly be summarized in one sentence: all natural laws must be so conditioned that they are covariant with respect to Lorentz transformations.

In fact, the precise relationship between the two conditions is much more complex. To see this relationship in more detail, we previously need to clarify a few things.

Consider the following two sets: $P(\mathcal{E}) = \{P(F)|F \in \mathcal{E}\}$ and $\Lambda(\mathcal{E}) = \{\Lambda(F)|F \in \mathcal{E}\}$. Since a system of equations can be identified with its set of solutions, $P(\mathcal{E}) \subset 2^{\Omega'}$ and $\Lambda(\mathcal{E}) \subset 2^{R'}$ can be regarded as two systems of equations for relations between $\xi'_1, \xi'_2, \ldots \xi'_n$. In the primed variables, $P(\mathcal{E})$ has "the same form" as $\mathcal{E}$. Nevertheless, it can be the case that $P(\mathcal{E})$ does not express a true physical law, in the sense that its solutions do not necessarily describe true physical situations. In contrast, $\Lambda(\mathcal{E})$ is nothing but $\mathcal{E}$ expressed in variables $\xi'_1, \xi'_2, \ldots \xi'_n$.

Now, covariance intuitively means that equations $\mathcal{E}$ "preserve their forms against the transformation Λ". That is, in terms of the formalism we developed:

$$\Lambda(\mathcal{E}) = P(\mathcal{E}) \tag{2.85}$$

or, equivalently,

$$P(\mathcal{E}) \subset 2^{R'} \text{ and } \mathcal{E} = \Lambda^{-1}(P(\mathcal{E})) \tag{2.86}$$

36. The first thing we have to make clear is that—even if we know or presume that it holds—covariance (2.86) is obviously *not sufficient* for the relativity principle (2.75). For, (2.86) only guarantees the invariance of the set of solutions, $\mathcal{E}$, against "the Lorentz boost" $\Lambda^{-1} \circ P$, but it says nothing about which solution of $\mathcal{E}$ corresponds to which solution. In contrast, it is the very essence of the statement of the relativity principle that $\Lambda^{-1}(P(F))$ is *the* solution that describes the same physical system exhibiting the same phenomenon as the one described by F, except that the system is in motion with velocity $\mathbf{V}$ relative to K. For example, the mere covariance of the physical laws only implies that the Lorentz contracted configuration of a solid rod is *one of the possible* configurations admitted by the laws of physics governing the rod's behavior. But it does not imply that this configuration is the one that constitutes the rod in motion with velocity $\mathbf{V}$ relative to K. It must be clear that the fact that

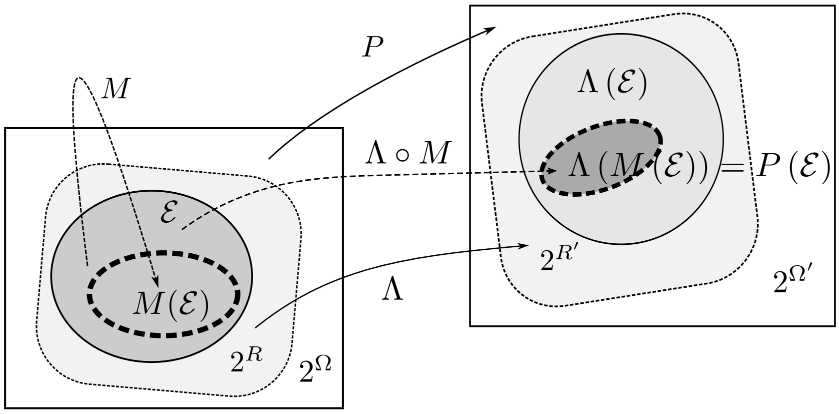

Figure 2.5: The relativity principle only implies that $\Lambda(\mathcal{E}) \supseteq \Lambda(M(\mathcal{E})) = P(\mathcal{E})$. Covariance of $\mathcal{E}$ would require that $\Lambda(\mathcal{E}) = P(\mathcal{E})$, which is generally not the case

covariance (2.86) does not imply the relativity principle (2.75) is simply a *logical* fact. This fact is prior to the physical problem of whether or not covariance is true; whether or not the relativity principle holds for a given physical situation; whether we have physically meaningful maps P_1,P_2; whether we know the transformation law Λ; whether we have an unambiguous meaning of $M(F)$, etc.

Let us note that, in a precise sense, covariance is not only not sufficient for the relativity principle, but it is *not even necessary* (Fig. 2.5). The relativity principle only implies that

$$\Lambda(\mathcal{E}) \supseteq \Lambda(M(\mathcal{E})) = P(\mathcal{E}) \tag{2.87}$$

Consequently, (2.74) implies (2.85) only if we have the following *additional* condition:

$$M(\mathcal{E}) = \mathcal{E} \tag{2.88}$$

37. It must be emphasized that when we claim that covariance is not equivalent with the principle of relativity, this statement has nothing to do with the extensively discussed problem of the relationship of general covariance and the relativity principle as it arises in the context of the geometric formulations of space-time theories (Friedman 1983; Earman and Norton 1987; Norton 1988, 1993; Brading and Brown 2004; Rovelli 2004; Dieks 2006; Westman and Sonego 2009). In particular, we do not intend to mean that the principle of covariance is physically vacuous. On the contrary, covariance (2.86) in itself has an inalterable physical meaning; in the sense that it states the invariance of the physical equations against transformation $\Lambda^{-1} \circ P$ the physical meaning of which is unambiguously determined by the physical meanings of Λ and P. As we pointed out in Paragraphs **21** and **24**, these laws, Λ and P, are determined by contingent physical facts concerning the behaviors of the measuring devices in different states of motion, by means of which the physical quantities $\xi_1, \xi_2, \ldots \xi_n$ and $\xi'_1, \xi'_2, \ldots \xi'_n$ are defined.

Once we understand the physical meaning of transformation $\Lambda^{-1} \circ P$ and, thereby, the physical meaning of covariance, we can think about the best mathematical representation of these physical facts—for example, in a Minkowski space, in terms of geometric transformations leaving the metric or other distinguished geometric object invariant. However, it must be emphasized, whether or not such a representation exits at all and what it actually looks like is a secondary question and does not in any way alter the meanings of the physical facts these mathematical structures are meant to express. One has to take this aspect into account especially with respect to the generalizations of the special principle of relativity (Grøn and Vøyenli 1999; cf. Friedman 1983, pp. 46–61).

38. Having the statements of the relativity principle and covariance expressed in terms of our formalism, it is now worth going back to the considerations of Section 2.1 and comparing these statements with other versions of the principle appearing at different points of the 1905 paper.

- According to the original understanding of the magnet-conductor thought experiment voiced by Einstein in the quoted passage in Paragraph **3**, it is stated that the current in the conductor only depends on the relative motion of the magnet and conductor, that is, it does *not* depend on the overall motion of the magnet-conductor system, relative to any given inertial frame K. What Einstein thus requires translates to

$$M(F) = F \tag{2.89}$$

  in our formalism; where F is the description of the system when, say, the magnet is rest and the conductor is in motion with velocity $\mathbf{V}$, and $M(F)$ is the description of the same behavior, but when the system, as a whole, is in motion with velocity $-\mathbf{V}$, that is the conductor is at rest and the magnet is in motion with velocity $-\mathbf{V}$, relative to K. Here the "description" of the system only extends to the current, that is, we have only one real valued quantity ξ corresponding to the value of current.

- In Griffiths's reading of the thought experiment (Paragraph **4**), in a *given* overall state of motion of the magnet-conductor system, the value of the current/electromotive force as described from *two different* inertial frames K and K' is the same. Let K' be in motion with velocity $\mathbf{V}$ relative to K, and keep the meaning of F as above; then Griffiths's condition translates to

$$\Lambda(F) = P(F) \tag{2.90}$$

That is to say, the description of this particular behavior of the system (the "equation" describing the value of current in the given state of the system) is required to be *covariant* (cf. (2.85)).

- In Paragraph **6** we formulated Einstein's requirement of covariance as two conditions. The first one demands that equations of the same form must *hold good* in any two inertial frames K and K'. Consider a set of equation $\mathcal{E}^* \subset 2^R$ not necessarily identical to $\mathcal{E}$ describing the system in question in K. We can say that $\mathcal{E}^*$ *holds good* in K if $\mathcal{E} \subseteq \mathcal{E}^*$, in other words, if in every physical situation described by an $F \in \mathcal{E}$ equations $\mathcal{E}^*$ are respected, that is $F \in \mathcal{E}^*$. Similarly, a set of equations $\mathcal{E}^{**} \subset 2^{R'}$ holds good in K' if $\Lambda(\mathcal{E}) \subseteq \mathcal{E}^{**}$. Now, with this, Einstein's first condition translates to

$$\mathcal{E} \subseteq \mathcal{E}^* \quad \text{if and only if} \quad \Lambda(\mathcal{E}) \subseteq P(\mathcal{E}^*) \tag{2.91}$$

  Observe that

  - (2.91) is not sufficient for $\mathcal{E}^*$ being covariant. It would require Einstein's second condition: $\mathcal{E}^*$ and $P(\mathcal{E}^*)$ must *express the same thing*, that is

$$\Lambda(\mathcal{E}^*) = P(\mathcal{E}^*) \tag{2.92}$$

  - Reversely, (2.92) obviously implies (2.91) due to $\Lambda(\mathcal{E}) \subseteq \Lambda(\mathcal{E}^*)$ if and only if $\mathcal{E} \subseteq \mathcal{E}^*$. In this sense, Einstein's two conditions of covariance reduces to the second one.
  - Applying (2.91) for $\mathcal{E}^* = \mathcal{E}$ and $P(\mathcal{E}^*) = \Lambda(\mathcal{E})$ one receives $\Lambda(\mathcal{E}) \subseteq P(\mathcal{E})$ and $P(\mathcal{E}) \subseteq \Lambda(\mathcal{E})$, respectively; that is, $\Lambda(\mathcal{E}) = P(\mathcal{E})$.
  - Reversely, the covariance of $\mathcal{E}$, (2.85), implies that (2.91) is true for all $\mathcal{E}^* \subset 2^R$. This is due to $P(\mathcal{E}) \subseteq P(\mathcal{E}^*)$ if and only if $\mathcal{E} \subseteq \mathcal{E}^*$. Thus, to say that equations of the same form hold good in any two inertial frames is equivalent to say that the equations actually describing the system in question are covariant.

Apparently, (2.89) and (2.90) are not equivalent with the relativity principle (2.74); recall that they are not even true in general (see Paragraph **3** and **4**). As (2.91) is equivalent with the covariance of $\mathcal{E}$, just as covariance it is also logically independent from the statement of the principle (see Paragraph **36**). On the other hand, if additional condition (2.88) is granted, then covariance (2.85) together with (2.91) follow from the relativity principle. It must be emphasized, however, that it is only the covariance of $\mathcal{E}$ which follows; that is, the covariance of that particular set of equations the solutions of which describe exactly the physically possible/realized behaviors of the system in question (cf. Paragraph **28**). With respect to any other set of physical equations $\mathcal{E}^*$, not its covariance, but only Einstein's first condition (2.91) is implied.

2.6 Initial and Boundary Conditions

39. Consider the situation when the solutions of a system of equations $\mathcal{E}$ are specified by some extra conditions—initial and/or boundary value conditions, for example. In our general formalism, an extra condition for $\mathcal{E}$ is a system of equations $\psi \subset 2^{\Omega}$ such that there exists exactly one solution $[\psi]_{\mathcal{E}}$ satisfying both $\mathcal{E}$ and ψ. That is, $\mathcal{E} \cap \psi = \{[\psi]_{\mathcal{E}}\}$, where $\{[\psi]_{\mathcal{E}}\}$ is a singleton set. Since $\mathcal{E} \subset 2^{R}$, without loss of generality we may assume that $\psi \subset 2^{R}$.

Since P and Λ are injective, $P(\psi)$ and $\Lambda(\psi)$ are extra conditions for equations $P(\mathcal{E})$ and $\Lambda(\mathcal{E})$ respectively, and we have

$$P([\psi]_{\mathcal{E}}) = [P(\psi)]_{P(\mathcal{E})} \tag{2.93}$$

$$\Lambda([\psi]_{\mathcal{E}}) = [\Lambda(\psi)]_{\Lambda(\mathcal{E})} \tag{2.94}$$

for all extra conditions ψ for $\mathcal{E}$. Similarly, if $P(\mathcal{E}), P(\psi) \subset 2^{R'}$ then $\Lambda^{-1}(P(\psi))$ is an extra condition for $\Lambda^{-1}(P(\mathcal{E}))$, and

$$\left[\Lambda^{-1}(P(\psi))\right]_{\Lambda^{-1}(P(\mathcal{E}))} = \Lambda^{-1}\left([P(\psi)]_{P(\mathcal{E})}\right) \tag{2.95}$$

If equations $\mathcal{E}$ satisfy the covariance condition (2.86), then $\Lambda^{-1}(P(\psi))$ is an extra condition for $\mathcal{E}$ and we have

$$\left[\Lambda^{-1}(P(\psi))\right]_{\mathcal{E}} = \Lambda^{-1}\left([P(\psi)]_{P(\mathcal{E})}\right) \tag{2.96}$$

That is to say, solving the primed equation with the primed extra conditions is equivalent to first expressing the primed extra conditions in the original quantities and then solving the original equations. Notice however that it by no means follows from *the covariance* of equations $\mathcal{E}$ that the primed extra conditions determine the solution describing the moving object; that is, it can be the case that $\left[\Lambda^{-1}\left(P\left(\psi\right)\right)\right]_{\mathcal{E}} \neq M\left(\left[\psi\right]_{\mathcal{E}}\right)$—this is the difference between the relativity principle and the covariance requirement.

40. Now consider a set of extra conditions $\mathcal{C} \subset 2^{2^R}$ and assume that $\mathcal{C}$ is a *parametrizing set of extra conditions* for $\mathcal{E}$; by which we mean that for all $F \in \mathcal{E}$ there exists exactly one $\psi \in \mathcal{C}$ such that $F = [\psi]_{\mathcal{E}}$; in other words,

$$\mathcal{C} \ni \psi \mapsto [\psi]_{\mathcal{E}} \in \mathcal{E} \tag{2.97}$$

is a bijection.

$M : \mathcal{E} \to \mathcal{E}$ was introduced as a map between solutions of $\mathcal{E}$. Now, as there is a one-to-one correspondence between the elements of $\mathcal{C}$ and $\mathcal{E}$, it generates a map $M : \mathcal{C} \to \mathcal{C}$, such that

$$[M(\psi)]_{\mathcal{E}} = M\left([\psi]_{\mathcal{E}}\right) \tag{2.98}$$

Thus, from (2.93) and (2.98), the relativity principle, that is (2.74), has the following form:

$$\Lambda\left([M(\psi)]_{\mathcal{E}}\right) = [P(\psi)]_{P(\mathcal{E})} \qquad \text{for all}\, \psi \in \mathcal{C} \tag{2.99}$$

or, equivalently, (2.75) reads

$$[P(\psi)]_{P(\mathcal{E})} \subset R' \;\text{ and }\; [M(\psi)]_{\mathcal{E}} = \Lambda^{-1}\left([P(\psi)]_{P(\mathcal{E})}\right) \tag{2.100}$$

41. If the covariance of equations $\mathcal{E}$ is guaranteed, the statement of the relativity principle can be expressed as a relation among the extra conditions for $\mathcal{E}$ or $P\left(\mathcal{E}\right)$.

Theorem 1. *Assume that the system of equations $\mathcal{E} \subset 2^R$ is covariant, that is, (2.85) is satisfied. Then,*

(i) for all $\psi \in \mathcal{C}$, $\Lambda\left(M\left(\psi\right)\right)$ is an extra condition for the system of equations $P\left(\mathcal{E}\right)$, and, (2.99) is equivalent to the following condition:

$$\left[\Lambda\left(M(\psi)\right)\right]_{P(\mathcal{E})} = [P(\psi)]_{P(\mathcal{E})} \tag{2.101}$$

(ii) for all $\psi \in \mathcal{C}, P\left(\psi\right) \subset 2^{R'}$, $\Lambda^{-1}\left(P\left(\psi\right)\right)$ is an extra condition for the system of equations $\mathcal{E}$ and (2.100) is equivalent to the following condition:

$$[M(\psi)]_{\mathcal{E}} = \left[\Lambda^{-1}\left(P\left(\psi\right)\right)\right]_{\mathcal{E}} \tag{2.102}$$

Proof. (i) Obviously, $\Lambda(\mathcal{E}) \cap \Lambda(M(\psi))$ exists and is a singleton; and, due to (2.85), it is equal to $P(\mathcal{E}) \cap \Lambda(M(\psi))$; therefore this latter is a singleton, too. Applying (2.94) and (2.85), we have

$$\Lambda([M(\psi)]_{\mathcal{E}}) = [\Lambda(M(\psi))]_{\Lambda(\mathcal{E})} = [\Lambda(M(\psi))]_{P(\mathcal{E})} \tag{2.103}$$

therefore, (2.101) implies (2.100).

(ii) Similarly, due to $P(\psi) \subset 2^{R'}$ and (2.86), $\mathcal{E} \cap \Lambda^{-1}(P(\psi))$ exists and is a singleton. Applying (2.95) and (2.86), we have

$$\Lambda^{-1}\left([P(\psi)]_{P(\mathcal{E})}\right) = \left[\Lambda^{-1}(P(\psi))\right]_{\Lambda^{-1}(P(\mathcal{E}))} = \left[\Lambda^{-1}(P(\psi))\right]_{\mathcal{E}} \tag{2.104}$$

that is, (2.102) implies (2.100). □

The equality of solutions in (2.101) and (2.102) do not warrant that the corresponding *extra conditions themselves*, $\Lambda(M(\psi))$ with $P(\psi)$, and $M(\psi)$ with $\Lambda^{-1}(P(\psi))$ respectively, are equal. This is because in general various different extra conditions may single out one and the same solution. However, consider the case when $\mathcal{C}$ is a *covariant* parametrizing set of extra conditions for $\mathcal{E}$; by which we mean that

$$\Lambda(\mathcal{C}) = P(\mathcal{C}) \tag{2.105}$$

or, equivalently,

$$P(\mathcal{C}) \subset 2^{2^{R'}} \text{ and } \mathcal{C} = \Lambda^{-1}(P(\mathcal{C})) \tag{2.106}$$

where $\Lambda(\mathcal{C}) = \{\Lambda(\psi) | \psi \in \mathcal{C}\}$ and $P(\mathcal{C}) = \{P(\psi) | \psi \in \mathcal{C}\}$. This implies the following.

Theorem 2. *Assume that the parametrizing set of extra conditions $\mathcal{C} \subset 2^{2^R}$ for $\mathcal{E}$ is covariant, that is, (2.105) is satisfied. Then,*

(i) for all $\psi \in \mathcal{C}$, condition (2.101) is equivalent with

$$\Lambda(M(\psi)) = P(\psi) \tag{2.107}$$

(ii) for all $\psi \in \mathcal{C}$, condition (2.102) is equivalent with

$$M(\psi) = \Lambda^{-1}(P(\psi)) \tag{2.108}$$

Proof. (2.105) implies that for all $\psi \in \mathcal{C}$, $\Lambda(M(\psi)) \in P(\mathcal{C})$ and $\Lambda^{-1}(P(\psi)) \in \mathcal{C}$. As $P(\psi) \in P(\mathcal{C})$, extra conditions $\Lambda(M(\psi))$ and $P(\psi)$ entering (2.101) belong to the same parametrizing set of extra conditions $P(\mathcal{C})$ for $P(\mathcal{E})$; and as $M(\psi) \in \mathcal{C}$, extra conditions $M(\psi)$ and $\Lambda^{-1}(P(\psi))$ entering (2.102) belong to the same parametrizing set of extra conditions $\mathcal{C}$ for $\mathcal{E}$. Since there is a one-to-one correspondence between solutions and extra conditions belonging to a parametrizing set, equalities of solutions (2.101) and (2.102) are equivalent with the equalities of the corresponding extra conditions themselves. Hence (2.107) and (2.108), respectively. □

Thus, given that the system of equations $\mathcal{E}$ as well as the the parametrizing set of extra conditions $\mathcal{C}$ by which we specify the solutions of $\mathcal{E}$ are *covariant*, the statement of the relativity principle can be expressed as the *additional* constraint of (2.107) or (2.108) imposed on the extra conditions.

42. To illustrate these concepts, let us continue the example of the two point masses connected by a spring in Paragraph **34**. The solutions to the equations of motion $\mathcal{E}$ given by (2.80) can be specified by initial value data of the following form:

$$\psi\left(x_{10}, x_{20}, v_{10}, v_{20}\right) = \left\{ \begin{array}{l} x_1(t=0) = x_{10} \\ x_2(t=0) = x_{20} \\ \left.\dfrac{dx_1}{dt}\right|_{t=0} = v_{10} \\ \left.\dfrac{dx_2}{dt}\right|_{t=0} = v_{20} \end{array} \right\} \tag{2.109}$$

To ensure $x_2(t) > x_1(t)$ for all t, the initial values $x_{10}, x_{20}, v_{10}, v_{20}$ must be chosen so that

$$x_{20} > x_{10} \tag{2.110}$$

and the center of mass energy of the system is less then $\frac{k}{2}L^2$, the center of mass energy when $x_2 = x_1$. The latter condition translates to

$$\frac{m}{4}\left(v_{20} - v_{10}\right)^2 + \frac{k}{2}\left(x_{20} - x_{10} - L\right)^2 < \frac{k}{2}L^2 \tag{2.111}$$

For example, solution F (2.81) corresponds to such an initial condition

$$\psi\left(x_{10} = -\frac{A+L}{2}, x_{20} = \frac{A+L}{2}, v_{10} = 0, v_{20} = 0\right) \tag{2.112}$$

with $A < L$.

Now, not only picks a ψ of form (2.109) a unique solution, but also different values for the initial data single out different solutions. So

$$\mathcal{C} = \left\{\psi\left(x_{10}, x_{20}, v_{10}, v_{20}\right) | x_{10}, x_{20}, v_{10}, v_{20} \text{ satisfy } (2.110)\text{–}(2.111)\right\} \tag{2.113}$$

is what we called a parametrizing set of extra conditions for equations $\mathcal{E}$. Moreover, $\mathcal{C}$ is covariant with respect the Galilean transformations (2.83), that is, (2.106) is satisfied. To see this, take an arbitrary $\psi(x_{10}, x_{20}, v_{10}, v_{20}) \in \mathcal{C}$. Form the same initial conditions as $\psi(x_{10}, x_{20}, v_{10}, v_{20})$ but in the primed variables:

$$P(\psi(x_{10}, x_{20}, v_{10}, v_{20})) = \left\{ \begin{array}{c} x'_1(t'=0) = x_{10} \\ x'_2(t'=0) = x_{20} \\ \left.\dfrac{dx'_1}{dt'}\right|_{t'=0} = v_{10} \\ \left.\dfrac{dx'_2}{dt'}\right|_{t'=0} = v_{20} \end{array} \right\} \tag{2.114}$$

Now eliminate the primes by means of the inverse of transformations (2.83):

$$\Lambda^{-1} \circ P(\psi(x_{10}, x_{20}, v_{10}, v_{20})) = \left\{ \begin{array}{c} x_1(t=0) = x_{10} \\ x_2(t=0) = x_{20} \\ \left.\dfrac{dx_1}{dt}\right|_{t=0} = v_{10} + V \\ \left.\dfrac{dx_2}{dt}\right|_{t=0} = v_{20} + V \end{array} \right\} \tag{2.115}$$

that is

$$\Lambda^{-1} \circ P(\psi(x_{10}, x_{20}, v_{10}, v_{20})) = \psi(x_{10}, x_{20}, v_{10} + V, v_{20} + V) \tag{2.116}$$

Since the new initial values $x_{10}, x_{20}, v_{10}+V, v_{20}+V$ also satisfy (2.110)–(2.111) as long as $x_{10}, x_{20}, v_{10}, v_{20}$ do so, $\Lambda^{-1} \circ P(\psi(x_{10}, x_{20}, v_{10}, v_{20})) \in \mathcal{C}$ and hence $\Lambda^{-1}(P(\mathcal{C})) \subseteq \mathcal{C}$. $\mathcal{C} \subseteq \Lambda^{-1}(P(\mathcal{C}))$ obtains in a similar way.

The equations of motion $\mathcal{E}$ are also covariant against Λ in the sense of (2.85). Indeed, expressing $\mathcal{E}$ in terms of the primed variables $(\xi'_1, \xi'_2, \ldots \xi'_6)$ through Λ one receives

$$\Lambda(\mathcal{E}) = \left\{ \begin{array}{c} m' \dfrac{d^2 x'_1(t')}{dt'^2} = k'(x'_2(t') - x'_1(t') - L') \\ m' \dfrac{d^2 x'_2(t')}{dt'^2} = -k'(x'_2(t') - x'_1(t') - L') \\ x'_2(t') > x'_1(t') \end{array} \right\} \tag{2.117}$$

which is manifestly identical with $P(\mathcal{E})$, that is, it has exactly the "same form" as the original equations $\mathcal{E}$.

Since both $\mathcal{E}$ and $\mathcal{C}$ are covariant one can apply the results of Paragraph **41** to express the relativity principle in terms of the elements of $\mathcal{C}$. Notice that $\psi\left(x_{10}, x_{20}, v_{10}+V, v_{20}+V\right)$ is nothing but $M\left(\psi\left(x_{10}, x_{20}, v_{10}, v_{20}\right)\right)$; for example the $M\left(F\right)$ given by (2.84) corresponds to initial condition

$$\psi\left(x_{10}=-\frac{A+L}{2}, x_{20}=\frac{A+L}{2}, v_{10}=V, v_{20}=V\right) \tag{2.118}$$

which indeed means just adding V to the initial velocities of initial condition (2.112) determining F. Thus, what (2.116) says is nothing but condition (2.107) which is equivalent with the statement of the principle according to Theorem 1 and 2. So, the relativity principle is not only satisfied by the particular solution (2.81), but, as (2.107) holds for all $\psi\left(x_{10}, x_{20}, v_{10}, v_{20}\right) \in \mathcal{C}$, it is also true for all $F \in \mathcal{E}$.

In contrast to the parametrizing set of extra conditions (2.113), initial value data of similar form as (2.109) will no longer be covariant for a relativistic system, such as the charge and its electromagnetic field. This is because the kinematic Lorentz transformation does not preserve the $t=0$ initial data surface, and hence for an initial condition ψ of form (2.109), $\Lambda^{-1}\circ P\left(\psi\right)$ will no longer even designate simultaneous data.

43. Let us note a few important—but often overlooked—facts which can easily be seen in the formalism we developed:

(a) The covariance of a set of equations $\mathcal{E}$ does *not* imply the covariance of a subset of equations separately. It is because a smaller set of equations corresponds to an $\mathcal{E}^* \subset 2^R$ such that $\mathcal{E} \subset \mathcal{E}^*$; and it does not follow from (2.85) that $\Lambda(\mathcal{E}^*) = P(\mathcal{E}^*)$.

(b) Similarly, the covariance of a set of equations $\mathcal{E}$ does *not* guarantee the covariance of an arbitrary set of equations which is only satisfactory to $\mathcal{E}$; for example, when the solutions of $\mathcal{E}$ are restricted by some further equations. Because from (2.85) it does not follow that $\Lambda(\mathcal{E}^*) = P(\mathcal{E}^*)$ for an arbitrary $\mathcal{E}^* \subset \mathcal{E}$.

(c) The same holds, of course, for the combination of cases (a) and (b); for example, when we have a smaller set of equations $\mathcal{E}^* \supset \mathcal{E}$ restricted by some other set of equations $\mathcal{E}^+ \subset 2^R$. For, (2.85) does not imply that $\Lambda(\mathcal{E}^* \cap \mathcal{E}^+) = P(\mathcal{E}^* \cap \mathcal{E}^+)$.

(d) However, covariance is guaranteed if a covariant set of equations is restricted with another *covariant* set of equations; because $\Lambda(\mathcal{E}) = P(\mathcal{E})$ and $\Lambda(\mathcal{E}^+) = P(\mathcal{E}^+)$ trivially imply that $\Lambda(\mathcal{E} \cap \mathcal{E}^+) = P(\mathcal{E} \cap \mathcal{E}^+)$.

44. As we have pointed out, covariance (2.86) expresses a symmetry property of the system of equations $\mathcal{E}$ as a whole, but it says nothing about the behavior of the moving objects. This is true even if covariance is sometimes formulated in terms of "processes corresponding to each other in different reference frames, described by the same functions, determined by the same initial conditions". We recall two examples:

(A)

> [I]f a possible process is described in the coordinates (x) by the functions
> $$\varphi_1(x), \varphi_2(x), \ldots \varphi_n(x) \tag{2.119}$$
> then there is another possible process which is describable by *the same* functions
> $$\varphi_1(x'), \varphi_2(x'), \ldots \varphi_n(x') \tag{2.120}$$
> in the coordinates (x'). Conversely any process of the form (2.120) in the second system corresponds to a possible process of the form (2.119) in the first system. (Fock 1964, p. 179)

(B)

> The simplest way to verify an invariance principle would be to create the same initial conditions in two equivalent coordinate systems and to observe whether the further fate of the two systems, from the point of view of the coordinate systems in question, is the same. (Houtappel, Van Dam, and Wigner 1965, p. 596)

Similarly to (A), of course, (B) is supposed to be true also with interchanging the roles of the two reference frames.

Now, in spite of the fact that these statements are about the "corresponding processes" (Fock) in different reference frames in relative motion, what they actually assert are *equivalent* to the covariance (2.85), but not to the relativity principle (2.74). In order to see that, we translate these assertions to the language we developed and prove the following theorem:

Theorem 3. *The following statements are equivalent to the condition that the set of equations $\mathcal{E}$ satisfies covariance (2.85):*

(A) *For all* $F \subset 2^R$ *and for all* $F' \subset 2^{R'}$,

$$F \in \mathcal{E} \quad \textit{implies} \quad P(F) \in \Lambda(\mathcal{E}) \tag{2.121}$$

$$F' \in \Lambda(\mathcal{E}) \quad \textit{implies} \quad P^{-1}(F') \in \mathcal{E} \tag{2.122}$$

(B) *There exists a parametrizing set of extra conditions* $\mathcal{C}$ *for the set of equations* $\mathcal{E}$*, such that for all* $\psi \in \mathcal{C}$*,* $P(\psi)$ *is an extra condition for the set of equations* $\Lambda(\mathcal{E})$ *and*

$$P\left([\psi]_{\mathcal{E}}\right) = [P(\psi)]_{\Lambda(\mathcal{E})} \tag{2.123}$$

and, there exists a parametrizing set of extra conditions $\mathcal{C}'$ *for the set of equations* $\Lambda(\mathcal{E})$*, such that for all* $\psi' \in \mathcal{C}'$*,* $P^{-1}(\psi')$ *is an extra condition for the set of equations* $\mathcal{E}$ *and*

$$P^{-1}\left([\psi']_{\Lambda(\mathcal{E})}\right) = \left[P^{-1}(\psi')\right]_{\mathcal{E}} \tag{2.124}$$

Proof. It is trivially true that (2.85) implies (2.121) and (2.122) for any $F \subset 2^R$ and $F' \subset 2^{R'}$. It also implies, for arbitrary extra conditions ψ and ψ', that $P(\psi)$ is an extra condition for $\Lambda(\mathcal{E})$ and $P^{-1}(\psi')$ is an extra condition for $\mathcal{E}$, and (2.123)–(2.124) are true—as we can see it from (2.93), (2.94), and (2.96).

As to the opposite direction, if (2.121) is true for all $F \subset 2^R$ then $P(\mathcal{E}) \subseteq \Lambda(\mathcal{E})$. On the other hand, if (2.122) is true for all $F' \subset 2^{R'}$, then $P^{-1}(\Lambda(\mathcal{E})) \subseteq \mathcal{E}$, therefore $\Lambda(\mathcal{E}) \subseteq P(\mathcal{E})$.

Similarly, if $P(\psi)$ is an extra condition for $\Lambda(\mathcal{E})$ and (2.123) is true for a set of extra conditions constituting a parametrizing set for $\mathcal{E}$, then $P(\mathcal{E}) \subseteq \Lambda(\mathcal{E})$. At the same time, if $P^{-1}(\psi')$ is an extra condition for $\mathcal{E}$ and (2.124) is true for a set of extra conditions constituting a parametrizing set for $\Lambda(\mathcal{E})$, then $\Lambda(\mathcal{E}) \subseteq P(\mathcal{E})$. □

Notice that if (2.123)–(2.124) are true for an arbitrary parametrizing set of extra conditions, then they are true for all possible extra conditions.

Again, the covariance of a system of equations only guarantees that every solution of the equations has a corresponding solution which has the same form in the other frame of reference; and the corresponding solution is determined by an extra condition of exactly the same form as the extra condition determining the original solution. Whether or not this corresponding solution is the one that describes the same phenomenon when the system in question, as a whole, is *in motion* is an additional fact of the physical world. This additional fact is stated by the relativity principle, over the covariance of the system of equations—see equation (2.102) in Theorem 1.

2.7 An Essential Conceptual Component: "Moving Body"

45. As we have seen, the principle of relativity does not reduce to the covariance of the physical equations, and the precise formulation of the principle is a much more difficult matter. It requires several conceptual plugins, without which the relativity principle would be simply meaningless. In Section 2.3 we gave the explicit formulation of these concepts in our formalism.

One of these concepts, $M : \mathcal{E} \to \mathcal{E}$, plays a crucial role. In accordance with the title of Einstein's 1905 paper, "On the Electrodynamics of Moving Bodies", the principle of relativity is about the comparison of the descriptions of moving physical systems in different inertial frames of reference moving relative to each other. In our formulation, the concept of moving systems essentially boils down to the notion of M. Thus, M carries an essential part of the physical content of the relativity principle; the precise meaning of the principle crucially depends on how M is physically understood. But what does it generally mean to say that a solution, $M(F)$, describes the same physical system exhibiting the same phenomenon as the one described by F, except that the system is in motion relative to K, with velocity $\mathbf{V}$, together with inertial frame K'?

In fact the same question can be asked with respect to the definitions of maps P_1 and P_2—and, therefore, with respect to the actual meanings of Λ and P. For, according to Paragraph **21**, $\xi'_1, \xi'_2, \ldots \xi'_n$ are not simply arbitrary variables assigned to reference frame K', in one-to-one relations with $\xi_1, \xi_2, \ldots \xi_n$, but the physical quantities obtainable by means of the same operations with the same measuring equipments as in the operational definitions of $\xi_1, \xi_2, \ldots \xi_n$, except that everything is in a collective motion with velocity $\mathbf{V}$. Therefore, we should know what we mean by "the same measuring equipment but in collective motion". From this point of view, it does not matter whether the system in question is the object to be observed or a measuring equipment involved in the observation.

These questions are to be answered within the concrete physical contexts. As we will see in Chapter 4, in some situations the answers are non-trivial and ambiguous. At this level of generality we only want to point out two things.

46. First, whatever is the definition of $M : \mathcal{E} \to \mathcal{E}$ in the given context, the following is a *minimum* requirement for the relativity principle to have the assumed physical meaning:

(M) Every relation $F \in \mathcal{E}$ must describe a phenomenon which can be mean-

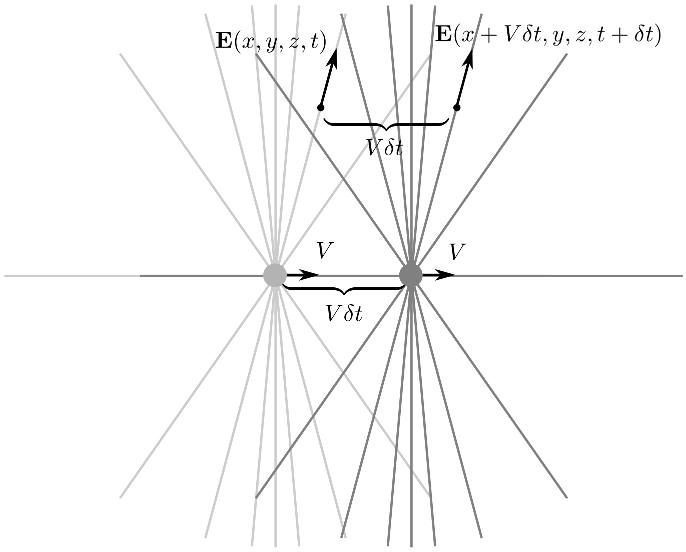

Figure 2.6: The stationary field of a uniformly moving point charge is in collective motion together with the point charge

> ingfully characterized as such that the physical system exhibiting this phenomenon is co-moving with some inertial frame of reference.

Recall that this minimum requirement is, tacitly, already there in Galileo's principle. As Brown points out:

> The process of putting the ship into motion corresponds [...] to what today we call an active pure boost of the laboratory. A key aspect of Galileo's principle that we wish to highlight is this. For Galileo, the boost is a clearly defined operation pertaining to a certain subsystem of the universe, namely the laboratory (the cabin and equipment contained in it). The principle compares the outcome of relevant processes inside the cabin under different states of inertial motion of the cabin relative to the shore. It is simply assumed by Galileo that the same initial conditions in the cabin can always be reproduced. What gives the relativity principle empirical content is the fact that the differing states of motion of the cabin are clearly distinguishable relative to the earth's rest frame. (Brown 2005, p. 34)

A simple example for a system satisfying condition (M) is the one discussed in Paragraph **33**: solutions (2.76) and (2.77) both describe a system of charged particle + electromagnetic field which are in collective rest and motion respectively. Indeed, the point charge and its electromagnetic field are in collective motion with velocity V ($V = 0$ corresponds to rest) along the x-axis (Fig. 2.6) in the following sense:

$$\mathbf{E}(x,y,z,t) = \mathbf{E}(x+V\delta t,y,z,t+\delta t) \tag{2.125}$$
$$\mathbf{B}(x,y,z,t) = \mathbf{B}(x+V\delta t,y,z,t+\delta t) \tag{2.126}$$
$$\frac{dr_x(t)}{dt} = V \tag{2.127}$$

But, generally, condition (M) by no means requires that the system be in a simple stationary state and all parts move with the same collective velocity—the objects contained in Galileo's cabin may exhibit very complex time-dependent behavior; the fishes may swim with their fins, the butterflies may move their wings, the particles of the smoke may follow a very complex dynamics. An example for such a more complicated behavior still satisfying (M) is provided by the case in Paragraph **34**: even tough the two point masses connected by the spring are in oscillatory motion with different, varying velocities, solutions (2.81) and (2.84) can both still qualify as descriptions of the system being in collective rest and motion respectively, as, for instance, the center of mass of the particles is at rest versus in motion with velocity V.

Notice that requirement (M) does not even say anything about whether and how the fact that the system in question is co-moving with a reference frame is reflected in a solution $F\in\mathcal{E}$. It does not even require that this fact can be expressed in terms of quantities $\xi_1,\xi_2,\ldots\xi_n$. It only requires that each $F\in\mathcal{E}$ belong to a physical situation in which it is meaningful to say—perhaps in terms of quantities different from $\xi_1,\xi_2,\ldots\xi_n$—that the system is at rest or in motion relative to an inertial reference frame. How a concrete physical situation can be so characterized is a separate problem, which can be discussed in the particular contexts. For example, even this minimum requirement can raise non-trivial questions in electrodynamics which will be the subject of Chapter 4.

47. The second thing to be said about $M(F)$ is that it is a notion determined by the concrete physical context; but it is *not* equal to the "Lorentz boosted solution" $\Lambda^{-1}(P(F))$ *by definition*—as the following reflections show:

(a) In this case, (2.75) would read

$$\Lambda^{-1}(P(F)) = \Lambda^{-1}(P(F)) \tag{2.128}$$

That is, the principle of relativity would become a tautology; a statement which is always true, independently of any contingent fact of nature; independently of the actual behavior of moving physical objects; and independently of the actual empirical meanings of physical quantities $\xi'_1,\xi'_2,\ldots\xi'_n$. This would contradict to the view—shared by a number of physicists and philosophers (see Brading and Castellani 2008)—that the statement of relativity/covariance principle, like many other symmetry principles, must be considered as a contingent, empirically falsifiable, statement. As Houtappel, Van Dam, and Wigner warn us:

> The discovery of Lee, Yang, and Wu, showing, among other facts, that the laws of nature are not invariant with respect to charge conjugation, reminded us of the empirical origin of the laws of invariance in a forcible manner. Before the discoveries of Lee, Yang and Wu, one could quote Fourier's principle as an earlier example of an invariance principle which had to be abandoned because of empirical evidence. (Houtappel, Van Dam, and Wigner 1963, p. 597)

Earman points out a more general epistemological aspect:

> [V]iewing symmetry principles as meta-laws doesn't commit one to treating them *a priori* in the sense of known to be true independently of experience. For instance, that a symmetry principle functions as a valid meta-law can be known *a posteriori* by a second level induction on the character of first-order law candidates that have passed empirical muster. (Earman 2004, p. 6)

Notice that even the transformation rules must be considered as empirically falsifiable laws of nature. For, how can we verify even a single instance of the covariance principle? One might think that the verification of the covariance of a given law of physics is only a matter of mathematical verification. But this is true only if we know the transformation laws of the physical quantities—against which the physical law in question must be covariant. Consequently, we must have an independent knowledge of the transformation rules expressible in terms of the physical behavior of the measuring equipments—in various states of motion—by means of which the physical quantities are operationally defined.[11] For, as Einstein emphasizes:

> A Priori it is quite clear that we must be able to learn something about the physical behavior of measuring-rods and clocks from the equations of transformation, for the magnitudes z, y, x, t are nothing more nor less than the results of measurements obtainable by means of measuring-rods and clocks. (Einstein 1920, p. 35)

[11] A case study illustrating this will be provided in Chapter 3.

Note that a tautology is entirely different from a fundamental principle, even if the principle is used as a fundamental hypothesis or fundamental premise of a theory, from which one derives further physical statements. For, a fundamental premise, as expressing a contingent fact of nature, is potentially falsifiable by testing its consequences; a tautology is not.

(b) Even if accepted, $M(F) \overset{def}{=} \Lambda^{-1}(P(F))$ can provide physical meaning to $M(F)$ only if we know the meanings of Λ and P, that is, if we know the empirical meanings of the quantities denoted by $\xi_1', \xi_2', \ldots \xi_n'$. But, the physical meaning of $\xi_1', \xi_2', \ldots \xi_n'$ are obtained from the physical meanings of the maps P_1 and P_2. But they are based on the concepts of the same measurement operations and the same measurement outcomes with the same equipments when they are co-moving with K' with velocity $\mathbf{V}$ relative to K. Symbolically, we need, priorly, the concepts of $M(\xi_i\text{-}equipment\,at\,rest)$. And this is a conceptual circularity: in order to have the concept of what it is to be an $M(brick\,at\,rest)$ the $(size)'$ of which we would like to ascertain, we need to have the concept of what it is to be an $M(measuring\,rod\,at\,rest)$—which is exactly the same conceptual problem. To put it slightly differently, the meanings of $\xi_1', \xi_2', \ldots \xi_n'$ as well as Λ and P presuppose the inertial frame K', the frame of reference in motion with velocity $\mathbf{V}$ relative to K. But how could we understand even the concept of "the reference frame in motion relative to K"—simbolically, $M(K)$—without the prior understanding of what M is?

(c) One might claim that we do not need to specify the concepts of $M(\xi_i\text{-}equipment\,at\,rest)$ or $M(K)$ in order to know the *values* of quantities $\xi_1', \xi_2', \ldots \xi_n'$ we obtain by the measurements with the moving equipments, given that we can know the transformation rule Λ independently of knowing the operational definitions of $\xi_1', \xi_2', \ldots \xi_n'$. Typically, Λ is thought to be derived from the assumption that the relativity principle (2.75) holds. If however M is, by definition, equal to $\Lambda^{-1} \circ P$, then in place of (2.75) we have the tautology (2.128), which does not determine Λ.

(d) Therefore, unsurprisingly, it is not the relativity principle from which the transformation rules are routinely deduced, but covariance (2.86). As we have seen, however, covariance is, in general, neither sufficient nor necessary for the principle. Whether (2.75) implies (2.86) hinges on the physical fact whether (2.88) is satisfied. But, if M is taken to be $\Lambda^{-1} \circ P$ by definition, the relativity principle becomes true—in the form of tautology (2.128)—but does not imply covariance $\Lambda^{-1}(P(\mathcal{E})) = \mathcal{E}$.

(e) Even if we assume that a "transformation rule" function $\phi' \circ \Lambda \circ \phi^{-1}$ were derived from some independent premises—typically from the independent assumption of covariance—how do we know that the Λ we obtained and the quantities of values $\phi' \circ \Lambda \circ \phi^{-1}(\xi_1, \xi_2, \ldots \xi_n)$ are correct plugins for the relativity principle? In a mathematical sense numeric equations $\phi(\mathcal{E})$ may have many different "symmetry transformations" $\phi' \circ \Lambda \circ \phi^{-1}$ satisfying "covariance" $\phi' \circ \Lambda \circ \phi^{-1}(\phi(\mathcal{E})) = \phi(\mathcal{E})$; which may not be in any way related to the quantities of the moving frame K'. How could we verify that $\phi' \circ \Lambda \circ \phi^{-1}(\xi_1, \xi_2, \ldots \xi_n)$ are indeed the values measured by a moving observer applying the same operations with the same measuring equipments, etc.?—without having an independent concept of M, at least for the measuring equipments?

(f) One could argue that we do not need such a verification; $\phi' \circ \Lambda \circ \phi^{-1}(\xi_1, \xi_2, \ldots \xi_n)$ can be regarded *as the empirical definition* of the primed quantities:

$$(\xi_1', \xi_2', \ldots \xi_n') \stackrel{def}{=} \phi' \circ \Lambda \circ \phi^{-1}(\xi_1, \xi_2, \ldots \xi_n) \tag{2.129}$$

where $\phi' \circ \Lambda \circ \phi^{-1}$ is derived from the assumed covariance of the equations in question. This is of course logically possible. The operational definition of the primed quantities would say: ask the observer at rest in K to perform the measurements $s_1, s_2, \ldots s_n$ with the equipments at rest in K, and then perform the mathematical operation (2.129) on the results $\xi_1, \xi_2, \ldots \xi_n$ so obtained. In this way, however, even the requirement of covariance would become tautological; for every equation is covariant against the transformation rules derived from the presumption that the equation itself is covariant.

(g) Someone might claim that the identity of M with $\Lambda^{-1} \circ P$ is not a simple stipulation but rather an analytic truth which follows from the identity of the two *concepts*. Still, if that were the case, the relativity principle would be a statement which is true in all possible worlds; independently of any contingent fact of nature; independently of the actual behavior of moving physical objects.

(h) On the contrary, the relationship between the Lorentz boosted solution and the one describing the actual motion of the system is a more sophisticated issue. As Bell points out in his famous two-spaceship paper:

> Lorentz invariance alone shows that for any state of a system at rest there is a corresponding 'primed' state of that system in motion. But it does not tell us that if the system is set anyhow in motion, it will actually go into the 'primed' of the original state, rather than

> into the 'prime' of some *other* state of the original system. In fact, it will generally do the latter. (Bell 1987, p. 75)

That is to say, in some situations, in spite of the fact that the physical laws in question are covariant, the Lorentz boosted solution $\Lambda^{-1}(P(F))$ is not identical with the one describing the system set in motion (also see Jánossy 1971, pp. 207–210; Szabó 2004). The mere conceivability of such a situation means that $M(F)$ and $\Lambda^{-1}(P(F))$ are different concepts.

(i) As we have already pointed out in Paragraph **32**, $M(F)$ and $\Lambda^{-1}(P(F))$ are concepts referring to *different* features of *different* parts of the physical reality. Any connection between the two things must be a contingent fact of the world. The map $\Lambda^{-1} \circ P$ is completely determined by the physical behaviors of the *measuring* equipments. Consequently, even if F is a description of a particular phenomenon exhibited by the object system, nothing guarantees that $\Lambda^{-1}(P(F))$ has anything to do with the behavior of the object system; the physical behaviors of the measuring equipments do not guarantee that $\Lambda^{-1}(P(\mathcal{E})) \subseteq \mathcal{E}$, nor that the elements of $\mathcal{E}$ satisfy condition (M). For example, from the information of how the static field of a charge at rest looks like—formula (2.62)—and how the transformation laws of electrodynamic quantities look like—regarded as independent empirical facts about the measuring equipments—one can determine the Lorentz boosted field (2.65), no matter how the system of equations of electrodynamics looks like, no matter whether (2.65) is a solution of these equations or not, and no matter whether this solution is the one describing the field of the uniformly *moving* charge.

(j) In the standard applications of the relativity principle, M is used as an independent concept, without any prior reference to the Lorentz boost $\Lambda^{-1} \circ P$. Continuing the above example, we do not need to refer to the transformations laws of the electrodynamic quantities in order to understand the concept of 'the electromagnetic field of a uniformly moving point charge'; as we are capable to describe this phenomenon by solving the electrodynamic equations for such a situation within one single frame of reference.

(k) Finally, we note that it makes no essential difference if $M(F)$ were defined as the solution describing "the process determined by the same initial state with respect to the second frame as the original system had with respect to the first", that is:

$$M(F) \stackrel{def}{=} \left[\Lambda^{-1}\left(P\left(\psi\right)\right)\right]_{\mathcal{E}} \text{ where } F = [\psi]_{\mathcal{E}} \tag{2.130}$$

The reason is that if this definition provides a well-defined $M(F)$, then it is equivalent to $M(F) \stackrel{def}{=} \Lambda^{-1}\left(P(F)\right)$. For (2.130) is meaningful only if it defines a unique $M(F)$ for any ψ satisfying $F = [\psi]_{\mathcal{E}}$. Then it must be so for $\psi = \{F\}$ too. In this case, however, $[\Lambda^{-1}\left(P\left(\{F\}\right)\right)]_{\mathcal{E}} = [\Lambda^{-1}\left(\{P\left(F\right)\}\right)]_{\mathcal{E}} = [\{\Lambda^{-1}\left(P\left(F\right)\right)\}]_{\mathcal{E}} = \Lambda^{-1}\left(P(F)\right)$.

Chapter 3

Operational Understanding of the Covariance of Electrodynamics

3.1 A Logical Problem with Postulating Relativity

48. In the previous chapter we clarified the *meaning* of the transformation laws of physical quantities, the covariance of physical equations and the requirement of the relativity principle. Now we turn to the question of how to *ascertain* what the transformation laws are and whether the relativity principle and covariance hold. This is, primarily, an *empirical* question (cf. Paragraph **47**, (a)).

(1) We need to compare the results of measurements performed in different inertial frames of reference, with measuring equipments co-moving relative to each other, and read off the functional relationships between them.

(2) Then, we need to check whether the physical equations describing the system in question in a given frame of reference, which, again, we learn from experience, satisfy covariance (2.85) against the transformation laws thus obtained.

(3) Finally, we need to verify whether the solutions of the equations describing the system in different states of motion satisfy the relativity principle (2.74) with the empirically ascertained transformation laws; or, equivalently, if the equations turned out to be covariant, whether the corresponding initial and boundary conditions satisfy (2.102).

However, it is hard to find any reference to such a verification of the facts of relativity. What we do find in most of the literature, instead, is an account inspired by the method of Einstein's 1905 paper. Here is Norton 1993 phrasing in expressive terms how we usually think about the way the relativity principle applies to electrodynamics as established by Einstein:

> Selecting suitable transformation laws for the field and other quantities, Einstein was able to show that the laws of electrodynamics remained unchanged under the Lorentz transformation. That is, they were Lorenz covariant. Therefore, within the space and time of special relativity, electrodynamics could no longer pick out any inertial frame of reference as preferred. Each inertial frame was fully equivalent within the laws of the theory. [...] Electrodynamics was now compatible with the relativity of inertial motion. (Norton 1993, p. 796)

It is interesting to find the following passage in the first edition of Jackson's *Classical Electrodynamics* where the author clearly indicates the possibility of an account of type step (1)–(2), but then dismisses it to follow Einstein's line:

> The invariance in form of the equations of electrodynamics under Lorentz transformations was shown by Lorentz and Poincaré before Einstein formulated the special theory of relativity. We will now discuss this covariance and consider its consequences. There are two points of view possible. One is to take some experimentally proven fact such as the invariance of electric charge and try to deduce that the equations must be covariant. The other is to demand that the equations be covariant in form and to show that the transformation properties of the various physical quantities, such as field strengths and charge and current, can be satisfactorily chosen to accomplish this. Although the first view is to some the most satisfying, we will adopt the second course. Classical electrodynamics is correct, and it can be cast in covariant form. (Jackson 1962, p. 377)

In latter editions of the book this passage is modified with no mention of the first possibility:

> The invariance in form of the equations of electrodynamics under Lorentz transformations was shown by Lorentz and Poincaré before the formulation the special theory of relativity. This invariance of form or *covariance* of the Maxwell and Lorentz force equation implies that the various quantities $\varrho, \mathbf{j}, \mathbf{E}, \mathbf{B}$ that enter these equations transform in well-defined ways under Lorentz transformations. Then the terms of the equations can

have consistent behavior under Lorentz transformations. (Jackson 1999, p. 553)

But there is a tension between the picture these quotes suggest and the way in which the relativity principle can be verified via steps (1)–(3). One obvious problem is, as discussed in Section 2.5, equating the relativity principle with covariance. Nevertheless, even if "the relativity of inertial motion" means the covariance of the physical laws, the words of Norton and Jackson reflect a seriously misleading idea. We are not entitled to "*select*" or "*satisfactorily choose*" the transformation laws for they are *determined by facts* about the physical behavior of the measuring equipments in different states of motion, in interaction with the measured physical reality; which we can, in principle, access empirically according to step (1). In other words, the covariance of a physical equation cannot be simply *"shown"* by being able to *select* the transformations of the quantities entering the equation so that the equation remains unchanged under these "well selected" transformations. The principle of covariance is not a matter of "casting the theory in covariant form" by means of "satisfactorily chosen" transformation rules, it is not a matter of "the terms of the equations having consistent behavior under Lorentz transformations". For, the laws of physics must be covariant against the *real physical* transformation laws linking the results of measurements in different frames (cf. Paragraph **37**). If the above accounts of what "compatibility with the relativity of inertial motion" consists in were correct, relativity would reduce to a tautology, a kind of analytic truth; for every equation is covariant against the transformation rules that are selected so that the equation be covariant (cf. Paragraph **47**, (e)–(f)).

49. To illustrate this problem, it is worth taking an example. Consider a scalar plane wave traveling along the x-axis of reference frame K with velocity c. The corresponding equation in K is

$$u\left(x,y,z,t\right)=A\sin\left(\omega\left(\frac{x}{c}-t\right)\right) \tag{3.1}$$

Suppose that this equation is required to be covariant, so that when expressing it in terms of the primed variables of a reference frame K' through the (yet unknown) transformation laws, we assume to receive

$$u'\left(x',y',z',t'\right)=A'\sin\left(\omega'\left(\frac{x'}{c'}-t'\right)\right) \tag{3.2}$$

This requirement is compatible with many different possible transformation rules. It is easy to check that equation (3.1) is covariant against, for example, a Galilean transformation

$$\Lambda^* = \left\{ \begin{array}{l} x' = x - Vt \\ y' = y \\ z' = z \\ t' = t \\ u' = u \\ A' = A \\ \omega' = \omega \left(1 - \dfrac{V}{c}\right) \\ c' = c - V \end{array} \right\} \tag{3.3}$$

or a Lorentz transformation

$$\Lambda^{**} = \left\{ \begin{array}{l} x' = \dfrac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}} \\ y' = y \\ z' = z \\ t' = \dfrac{t - \frac{V}{c^2}x}{\sqrt{1 - \frac{V^2}{c^2}}} \\ u' = u \\ A' = A \\ \omega' = \omega \sqrt{\dfrac{1 - \frac{V}{c}}{1 + \frac{V}{c}}} \\ c' = c \end{array} \right\} \tag{3.4}$$

or a Lorentz transformation with a non-invariant, contracted amplitude

$$\Lambda^{***} = \left\{ \begin{aligned} x' &= \frac{x - Vt}{\sqrt{1-\frac{V^2}{c^2}}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - \frac{V}{c^2}x}{\sqrt{1-\frac{V^2}{c^2}}} \\ u' &= u\sqrt{1-\frac{V^2}{c^2}} \\ A' &= A\sqrt{1-\frac{V^2}{c^2}} \\ \omega' &= \omega\sqrt{\frac{1-\frac{V}{c}}{1+\frac{V}{c}}} \\ c' &= c \end{aligned} \right\} \tag{3.5}$$

All this is a matter of mathematical fact. An equation like (3.1) may have many different symmetry transformations. And we are certainly able to derive them from the *assumption* that the equation is covariant. But when we ask whether the *physical* transformations, the ones connecting the results of measurements in K and K', are among the transformations so derived, and thus whether the equation is covariant against those *physical* transformations, to these question such a derivation gives no answer. Notice that at least two of the transformations $\Lambda^*, \Lambda^{**}, \Lambda^{***}$ we can obtain *by this derivation* will have nothing to do with the real transformations; and hence their being symmetries of equation (3.1) will have nothing to do with whether or not the principle of covariance is satisfied. How do we know which is the right transformation and whether it is among them at all?

Of course the mathematical situation can be more complex: one may demand the covariance of multiple equations at the same time, with the same, perhaps only few variables in them; one may have further physical assumptions about the form of the transformations, for example about their smoothness, or linearity, or dependence on certain physical parameters, etc; one may even know, priorly, how certain parts of the system of transformations actually look like. With all these further constraints, it may be a non-trivial mathematical question whether the covariance of the equations can be satisfied at all by selecting suitable transformations. Nevertheless, having this question answered, one can only say this much: If the answer is negative, then whatever the physical transformations are (as long as they satisfy all the constraints imposed), the equations in question cannot be covariant against them, and thus the principle of covariance is violated. If, however, one is able to find a suitable transformation Λ under which the equations are covariant, even if it is a unique one, this mathematical fact *gives no information whatsoever.* In particular, it does *not* follow 1) whether or not Λ is identical with the *real physical* transformations, and hence 2) whether or not against the *real physical* transformations the equations are covariant and thus the principle of covariance is satisfied. To ascertain these, *we need to have an independent knowledge of what the transformation laws are.*

50. It must be emphasized that the problem does not disappear even if the derivation of the transformation rules sometimes proceeds through casting the physical equation in "manifestly covariant", Lorentz tensorial form, by suitably identifying physical quantities as components of Lorentz tensors. Jackson writes:

> From the first postulate [the relativity principle] it follows that the mathematical equations expressing the laws of nature must be *covariant*, that is, invariant in form, under the transformations of the Lorentz Group. They must therefore be relations among Lorentz scalars, 4-vectors, 4-tensors, etc., defined by their transformation properties under the Lorentz group in ways analogous to the familiar specification of tensors of a given rank under three-dimensional rotations. (Jackson 1999, p. 540)

However, this is not supposed to mean that we are entitled to identify physical quantities as components of Lorentz tensors *in an arbitrary manner,* in order for the equations in question to look like relations among Lorentz scalars, 4-vectors, 4-tensors, etc., as the following passage from Jackson suggests:

> We begin with the charge density $\varrho(\mathbf{r}, t)$ and current density $\mathbf{j}(\mathbf{r}, t)$ and the continuity equation
>
> $$\frac{\partial \varrho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{3.6}$$
>
> From the discussion [about the transformation properties of the derivative operators with respect to x, y, z, t] and especially [from the fact that the 4-divergence of a 4-vector A, $\partial_\alpha A^\alpha = \partial^\alpha A_\alpha = \partial A^0/\partial x^0 + \nabla \cdot \mathbf{A}$, is invariant], it is natural to postulate that ϱ and $\mathbf{j}$ together form a 4-vector J^α:
>
> $$J^\alpha = (c\varrho, \mathbf{j}) \tag{3.7}$$

Then the continuity equation (3.6) takes the obviously covariant form,

$$\partial_\alpha J^\alpha = 0 \tag{3.8}$$

(Jackson 1999, p. 554–555)

On the contrary, whether $(c\varrho, \mathbf{j})$ *indeed* transforms as a 4-vector, and hence whether the continuity equation (3.6) is *indeed* covariant, depends on the form of the *physical* transformation laws of ϱ and $\mathbf{j}$ determined by contingent facts about the behavior of the measuring equipments with which the source densities are measured.

In fact, the representability of physical transformations in Lorentz tensorial form is not as straightforward as one might think. To see this, consider the general case of the transformations of electrodynamic quantities. Let K and K' be two inertial frames of reference. $x, y, z, t, \mathbf{E}, \mathbf{B}, \varrho, \mathbf{j}$ denote the basic physical quantities involved in electrodynamics, that is the space and time coordinates, the electric and magnetic field strengths, and the source densities, obtainable by means of measuring equipments co-moving with K. $x', y', z', t', \mathbf{E}', \mathbf{B}', \varrho', \mathbf{j}'$ are the same quantities in K', that is, the quantities obtainable by means of the same operations with the same measuring equipments when they are co-moving with K'. According to the general definition (2.68), the transformation law is a one-to-one functional relation,

$$\Lambda : (x, y, z, t, \mathbf{E}, \mathbf{B}, \varrho, \mathbf{j}) \mapsto (x', y', z', t', \mathbf{E}', \mathbf{B}', \varrho', \mathbf{j}') = \Lambda\left(x, y, z, t, \mathbf{E}, \mathbf{B}, \varrho, \mathbf{j}\right) \tag{3.9}$$

expressing the law-like regularity that if in an arbitrary space-time point A the K-quantities take values $(x(A), y(A), z(A), t(A), \mathbf{E}(A), \mathbf{B}(A), \varrho(A), \mathbf{j}(A))$ then, in the same space-time point A, the corresponding K'-quantities take values

$$\begin{aligned} &(x'(A), y'(A), z'(A), t'(A), \mathbf{E}'(A), \mathbf{B}'(A), \varrho'(A), \mathbf{j}'(A)) \\ &= \Lambda\left(x(A), y(A), z(A), t(A), \mathbf{E}(A), \mathbf{B}(A), \varrho(A), \mathbf{j}(A)\right) \end{aligned} \tag{3.10}$$

and vice versa.

Now, with regard to the form of this law, the following remarks are in order:

(a) First of all, note that, as emphasized in Paragraph **24**, one cannot a priori assume that there exists a transformation law in the form of (3.10); the fact that there is a law-like connection between the quantities in K and in K' at all is a contingent fact of the physical world.

(b) In order to be able to speak about the "Lorentz transformation" at all, transformation law Λ should split such that the space-time coordinates

$(x'(A), y'(A), z'(A), t'(A))$ are completely determined by the space-time coordinates $(x(A), y(A), z(A), t(A))$, without depending on the field strengths and the source densities $(\mathbf{E}(A), \mathbf{B}(A), \varrho(A), \mathbf{j}(A))$. That is, as a matter of further contingent fact, there must exist a map Λ_1 such that

$$(x'(A), y'(A), z'(A), t'(A)) \quad = \quad \Lambda_1 \left(x(A), y(A), z(A), t(A)\right) \tag{3.11}$$

The form of Λ_1 reflects the physical behavior of rods, clocks and light rays by means of which the space-time coordinates are ascertained.[1]

(c) If, further, the separation of Λ happens to be such that the field strengths and the source densities $(\mathbf{E}'(A), \mathbf{B}'(A), \varrho'(A), \mathbf{j}'(A))$ are completely determined by the field strengths and the source densities $(\mathbf{E}(A), \mathbf{B}(A), \varrho(A), \mathbf{j}(A))$, without depending on the space-time coordinates $(x(A), y(A), z(A), t(A))$, then this implies the existence of another map Λ_2:

$$(\mathbf{E}'(A), \mathbf{B}'(A), \varrho'(A), \mathbf{j}'(A)) \quad = \quad \Lambda_2 \left(\mathbf{E}(A), \mathbf{B}(A), \varrho(A), \mathbf{j}(A)\right) \tag{3.12}$$

In this case it is meaningful to talk about how the field strengths and the source densities "transform under the Lorentz transformation"; in the sense that the group theoretic structure of Λ_2, for varying K and K',[2] will be, as a matter of logical fact, homomorphic to that of Λ_1, and thus Λ_2 can be construed as a group "action" or "representation" of Λ_1.

(d) It must be emphasized, however, that the transformation of the space-time coordinates does *not* determine the transformations of the field strengths and the source densities—the Lorentz group has various different "actions". Now, which "action" is the one that is identical to Λ_2 is determined by contingent facts of the world. Therefore, one cannot a priori assume that

- Λ_2 is linear
- even if it is linear, it can be represented as a tensorial transformation with respect to the Lorentz transformation; since, for example, Λ_2 may easily contain physical parameters different from V and c (characterizing the physical behavior of the measuring equipments) and hence it may not be expressible in terms of the components of a Lorentz transformation

[1] In fact, to specify the precise, non-circular operational definitions of the space-time coordinates in a given frame of reference is not an obvious problem (see Szabó 2010).

[2] If Λ_2 exits at all for varying K and K'.

- Λ_2 is separable into two maps:

$$\left(\mathbf{E}'(A), \mathbf{B}'(A)\right) = \Lambda_2^*\left(\mathbf{E}(A), \mathbf{B}(A)\right) \tag{3.13}$$

$$\left(\varrho'(A), \mathbf{j}'(A)\right) = \Lambda_2^{**}\left(\varrho(A), \mathbf{j}(A)\right) \tag{3.14}$$

Whether these properties hold or not depends on the particular form of the *physical* transformation law Λ_2 determined by *physical facts* about the behavior of the measuring equipments by means of which the electrodynamic quantities are measured when set in different states of motion.

51. One might think that the validity of the standard derivations of transformation rules, either in the form of Einstein's algebraic considerations or with invoking the more advanced methods of tensor calculus, can be verified in a way different from directly verifying the derived transformations; in the sense of hypothetico-deductive confirmation, by means of empirical confirmation of the logical *consequences* of the hypothesis that the relativity principle/covariance as well as the derived transformation laws themselves hold. Such an understanding would probably be in line with Einstein's own views. Now, what empirically testable consequences one may have in mind? As is emphasized by Einstein himself in the 1905 paper, the relativity principle provides a powerful method for the physics of moving bodies (Einstein 1905, p. 59). The electromagnetic field of a moving point charge, the Lorentz deformation of a rigid body, the loss of phase suffered by a moving clock, the dilatation of the mean life of a cosmic ray μ-meson, the increase of mass of a moving charged particle—all these phenomena, which follow from the theory of special relativity as a whole, are in principle empirically testable or have been actually tested.

So, instead of steps (1)–(3) in Paragraph 48, one may suggest the following epistemological schema:

(1′) *Assume* that the equations of physics are covariant.

(2′) Derive the transformation laws from the assumption of covariance according to the standard procedures.

(3′) *Assume* that the relativity principle is true; then the behavior of moving physical systems can be obtained through the derived transformation laws according to $M(F) = \Lambda^{-1}\left(P(F)\right)$.

(4′) Verify empirically whether the description $M(F)$ so obtained is indeed identical to the observed behavior of the system set in motion.

In principle, this would be an acceptable way of confirming the hypotheses of a theory. However, as it will be argued in detail in Chapter 4, the method for obtaining the behavior of moving physical systems in step (3′) generally does not lead to the correct description in a relativistic setting.

52. So, how are we to verify the basic principles of relativity? More precisely, how do we know the answer to the following questions:

(Q1) Is the principle of relativity a true law of nature?

(Q2) Are the transformation rules of the fundamental physical quantities derived from the covariance assumption, and thereby the principle of covariance itself, true?

As emphasized in Paragraph **48**, this is, primarily, an empirical matter. However, we shall now outline an idea by means of which these questions can, in a natural way, be addressed in a theoretical framework.

3.2 Lorentzian Pedagogy

53. The basic idea is what J. S. Bell (1987, p. 77) calls "Lorentzian pedagogy", according to which "the laws of physics in any *one* reference frame account for all physical phenomena, including the observations of moving observers". That is to say, if the laws of physics that are valid in any one reference frame, say K, provide a complete enough description of the world, then they must account for the behaviors of the moving measuring equipments and the results of all measuring operations performed by a moving observer. Thus, the laws of physics in K determine the answers to questions (Q1) and (Q2); at least in the sense of what the answers are in the prediction of these laws. In this way, (Q1) and (Q2) naturally translate to the following theoretical questions:

(Q3) Is the principle of relativity consistent with the laws of physics in a single inertial frame of reference?

(Q4) Are the transformation rules derived from the covariance assumption, and thereby the principle of covariance itself, consistent with the laws of physics in a single inertial frame of reference?

54. It must be emphasized that adopting the Lorentzian pedagogy, construed as an inner consistency check of our theoretical hypotheses, by no means imply a commitment neither to an aether-theoretic framework, to a privileged reference frame, nor to a Lorentzian position in the methodological debate of Lorentz versus Einstein, in the sense of the *constructive* versus *principle* theory distinction (Bell 1992; Brown 2005). For:

1. The "priviliged" reference frame K can be an *arbitrary* frame of reference in which the laws of physics are assumed to be hold. No assumption about its "absolute rest" or similar notions is made (cf. Szabó 2011).

2. Further, no assumption is made about the nature of the physical laws in K; that is, whether they are counted as "detailed dynamical descriptions" or "principles"—as Bell expresses the difference between the *constructive* and *principle* accounts:

> If you are, for example, quite convinced of the second law of thermodynamics, of the increase of entropy, there are many things that you can get directly from the second law which are very difficult to get directly from a detailed study of the kinetic theory of gases, but you have no excuse for not looking at the kinetic theory of gases to see how the increase of entropy actually comes about. In the same way, although Einstein's theory of special relativity would lead you to expect the FitzGerald contraction, you are not excused from seeing how the detailed dynamics of the system also leads to the FitzGerald contractions. (Bell 1992, p. 34)

In contrast, the point of the Lorentzian pedagogy is that the physical content of the relativity principle/covariance is such that it characterizes the laws of physics in *one single* frame of reference. Therefore, whether these principles hold is not logically independent from how the laws of physics in one single frame look like. This is the reason why "we are not excused" from seeing how the laws of physics in one frame entail these principles.

3.3 The Case of Electrodynamics

55. Following Einstein's path, in the literature on classical electrodynamics and relativity theory the transformation laws of the basic electrodynamic quantities Λ_2 in (3.12) are derived from the assumption that Maxwell's equations are covariant against these transformation laws—in conjunction with the Lorentz transformation Λ_1. Among those with which we are acquainted, there are basically two major versions of these derivations, which are briefly summarized in the Appendix II. There are several problems to be raised concerning these derivations, and certain steps are questionable. This is however not our main concern in here. Again, even if these derivations are regarded as unquestionable, they only prove what the transformation laws Λ_2 *should* look like in order that Maxwell's equations constitute a covariant system of equations with respect to these transformations. But they leave open question (Q2).

As a case study for the Lorentzian pedagogy, the remainder of this chapter is devoted to answer the corresponding theoretical question (Q4) for the case of electrodynamics. That is, we shall be concerned with the following question:

(Q) How does the transformation laws Λ_2 in (3.12) look like in the prediction of the laws of electrodynamics in a single inertial frame of reference K?

As for the space-time coordinates, we will take it for granted that the functional relation Λ_1 in (3.11) is the Lorentz transformation.

The answer to (Q) can be given by the laws of physics only if the question is properly formulated. We must clarify what measuring equipments and etalons are used in the empirical definitions of the electrodynamic quantities; and we must be able to tell when two measuring equipments are the same, except that they are moving, as a whole, relative to each other—one is at rest relative to K, the other is at rest relative to another inertial frame K'. Similarly, we must be able to tell when two operational procedures performed by the two observers are the "same", in spite of the prima facie fact that the procedure performed in K' obviously differs from the one performed in K. In order to compare these procedures, first of all, we must know what the procedures exactly are. All in all, a correct answer to question (Q) can be given only on the bases of *a coherent system of precise operational definitions* of the quantities in question; and all these definitions must be represented in the language of electrodynamics in a single frame of reference. Interestingly, there is no explicit discussion of these issues in the standard literature on electrodynamics and special relativity; although, as we will see, none of these issues are as trivial as one might think.

Thus, accordingly, in the first part of the chapter we clarify the operational definitions of the electrodynamic quantities and formulate what electrodynamics in a single inertial frame of reference—let us call it "rest" frame—exactly asserts in terms of the quantities so defined. In the second part, applying the Lorentzian pedagogy, on the basis of the laws of electrodynamics in the "rest" frame, we derive what a moving observer must see in terms of the "rest" frame quantities when repeats the same operational procedures in the "moving" frame. In this way, we obtain the transformation laws Λ_2 of the electrodynamic quantities; that is to say, we derive the transformation laws from the precise operational definitions of the quantities and from the laws of electrodynamics in a single inertial frame of reference, *without of the pre-assumption that the equations are covariant* against these transformation laws—by which we answer our question (Q).

56. In Appendix I we summarize the most important formulas of standard Lorentzian kinematics, which we use in the calculations. Throughout we restrict ourselves to the traditional 3+1 formulation of classical electrodynamics. This is because the 4-dimensional formulation is *based* on the fact that the electrodynamic quantities transform in a certain way, while it is the very essence of this chapter to *derive* the transformation rules from the operational definitions of the quantities and from the laws of electrodynamics. In any case, the problem of ascertaining the true transformation laws of the electrodynamic quantities in any one empirically verifiable form is epistemologically *prior* to the problem of the algebraic/geometric formulation of these transformation laws. For, once we *know* these laws in any one available form, we can think about the best mathematical representation of them. (For a current discussion of the various mathematical formulations, see Ivezić 2001, 2003; Hestenes 1966, 2003; Huang 2008, 2009; Arthur 2011.)

3.4 Operational Definitions of Electrodynamic Quantities in K

57. In this section we give the operational definitions of the fundamental quantities of electrodynamics in a single reference frame K and formulate a few basic observational facts about these quantities.

The operational definition of a physical quantity requires the specification of *etalon* physical objects and standard physical processes by means of which the value of the quantity is ascertained. In case of electrodynamic quantities the only "device" we need is a point-like test particle, and the standard measuring procedures by which the kinematic properties of the test particle are ascertained.

So, assume we have chosen an *etalon* test particle, and let $\mathbf{r}^{etalon}(t)$, $\mathbf{v}^{etalon}(t)$, $\mathbf{a}^{etalon}(t)$ denote its position, velocity and acceleration at time t. It is assumed that we are able to set the *etalon* test particle into motion with arbitrary velocity $\mathbf{v}^{etalon} < c$ at arbitrary location. We will need more "copies" of the *etalon* test particle:

Definition (D0) A particle e is called *test particle* if for all $\mathbf{r}$ and t

$$\mathbf{v}^{e}(t)\Big|_{\mathbf{r}^{e}(t)=\mathbf{r}} = \mathbf{v}^{etalon}(t)\Big|_{\mathbf{r}^{etalon}(t)=\mathbf{r}} \tag{3.15}$$

implies

$$\mathbf{a}^{e}(t)\Big|_{\mathbf{r}^{e}(t)=\mathbf{r}} = \mathbf{a}^{etalon}(t)\Big|_{\mathbf{r}^{etalon}(t)=\mathbf{r}} \tag{3.16}$$

(The "restriction signs" refer to *physical* situations; for example, $|_{\mathbf{r}^{e}(t)=\mathbf{r}}$ indicates that the test particle e is at point $\mathbf{r}$ at time t.)

Note, that some of the definitions and statements below require the existence of many test particles; which is, of course, a matter of empirical fact, and will be provided by (E0) below.

58. First we define the electric and magnetic field strengths. The only measuring device we need is a test particle being at rest relative to K.

Definition (D1) *Electric field strength* at point $\mathbf{r}$ and time t is defined as the acceleration of an arbitrary test particle e, such that $\mathbf{r}^{e}(t)=\mathbf{r}$ and $\mathbf{v}^{e}(t)=0$:

$$\mathbf{E}(\mathbf{r},t) \stackrel{def}{=} \mathbf{a}^{e}(t)|_{\mathbf{r}^{e}(t)=\mathbf{r};\,\mathbf{v}^{e}(t)=0} \tag{3.17}$$

Magnetic field strength is defined by means of how the acceleration $\mathbf{a}^{e}$ of the rest test particle changes with an infinitesimal perturbation of its state of rest, that is, if an infinitesimally small velocity $\mathbf{v}^{e}$ is imparted to the particle. Of course, we cannot perform various small perturbations simultaneously on one and the same rest test particle, therefore we perform the measurements on many rest test particles with various small perturbations. Let $\delta \subset \mathbb{R}^{3}$ be an arbitrary infinitesimal neighborhood of $0 \in \mathbb{R}^{3}$. First we define the following function:

$$\begin{aligned}\mathbf{U}^{\mathbf{r},t} \;:\; & \mathbb{R}^{3} \supset \delta \to \mathbb{R}^{3} \\ & \mathbf{U}^{\mathbf{r},t}(\mathbf{v}) \stackrel{def}{=} \mathbf{a}^{e}(t)|_{\mathbf{r}^{e}(t)=\mathbf{r};\,\mathbf{v}^{e}(t)=\mathbf{v}}\end{aligned} \tag{3.18}$$

Obviously, $\mathbf{U}^{\mathbf{r},t}(0)=\mathbf{E}(\mathbf{r},t)$.

Definition (D2) *Magnetic field strength* at point $\mathbf{r}$ and time t is

$$\mathbf{B}(\mathbf{r},t) \stackrel{def}{=} \begin{pmatrix} \left.\frac{\partial U_y^{\mathbf{r},t}}{\partial v_z}\right|_{\mathbf{v}=0} \\ \\ \left.\frac{\partial U_z^{\mathbf{r},t}}{\partial v_x}\right|_{\mathbf{v}=0} \\ \\ \left.\frac{\partial U_x^{\mathbf{r},t}}{\partial v_y}\right|_{\mathbf{v}=0} \end{pmatrix} \tag{3.19}$$

Practically it means that one can determine the value of $\mathbf{B}(\mathbf{r},t)$, with arbitrary precision, by means of measuring the accelerations of a few test particles of velocity $\mathbf{v}^e \in \delta$.

59. Next we introduce the concepts of source densities:

Definition (D3)

$$\varrho(\mathbf{r},t) \stackrel{def}{=} \nabla \cdot \mathbf{E}(\mathbf{r},t) \tag{3.20}$$

$$\mathbf{j}(\mathbf{r},t) \stackrel{def}{=} c^2\nabla \times \mathbf{B}(\mathbf{r},t) - \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} \tag{3.21}$$

are called *active electric charge density* and *active electric current density,* respectively.

A simple consequence of the *definitions* is that a continuity equation holds for ϱ and $\mathbf{j}$:

Theorem 4.

$$\frac{\partial \varrho(\mathbf{r},t)}{\partial t} + \nabla \cdot \mathbf{j}(\mathbf{r},t) = 0 \tag{3.22}$$

60. In our construction, the two Maxwell equations (3.20)–(3.21), are mere *definitions* of the concepts of active electric charge density and active electric current density. They do not contain information whatsoever about how "matter produces electromagnetic field". And it is not because $\varrho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ are, of course, "unspecified distributions" in these "general laws", but because $\varrho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ cannot be specified prior to or at least independently of the field strengths $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. Again, because $\varrho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ are just abbreviations, standing for the expressions on the right hand sides of (3.20)–(3.21). In other words, any statement about the "charge distribution" will be a statement about $\nabla \cdot \mathbf{E}$, and any statement about the "current distribution" will be a statement about $c^2\nabla \times \mathbf{B} - \partial\mathbf{E}/\partial t$.

The minimal claim is that this is a possible coherent construction. Though we must add: equations (3.20)–(3.21) could be seen as contingent physical laws about the relationship between the charge and current distributions and the electromagnetic field, only if we had an independent empirical definition of charge. However, we do not see how such a definition is possible, without encountering circularities. (Also see Paragraph **62**)

61. The operational definitions of the field strengths and the source densities are based on the kinematic properties of the test particles. The following definition describes the concept of a charged point-like particle, in general.

Definition (D4) A particle b is called *charged point-particle* of *specific passive electric charge* π^b and of *active electric charge* α^b if the following is true:

1. It satisfies the relativistic Lorentz equation,

$$\begin{aligned} \frac{\mathbf{a}^b(t)}{\sqrt{1-\frac{\left(\mathbf{v}^b(t)\right)^2}{c^2}}} &= \pi^b \left\{ \mathbf{E}\left(\mathbf{r}^b(t), t\right) + \mathbf{v}^b(t) \times \mathbf{B}\left(\mathbf{r}^b(t), t\right) \right. \\ &\quad \left. -\frac{1}{c^2}\mathbf{v}^b(t)\left(\mathbf{v}^b(t)\cdot\mathbf{E}\left(\mathbf{r}^b(t), t\right)\right)\right\} \end{aligned} \tag{3.23}$$

2. If it is the only particle whose worldline intersects a given space-time region Λ, then for all $(\mathbf{r}, t) \in \Lambda$ the source densities are of the following form:

$$\varrho(\mathbf{r}, t) = \alpha^b \delta\left(\mathbf{r} - \mathbf{r}^b(t)\right) \tag{3.24}$$

$$\mathbf{j}(\mathbf{r}, t) = \alpha^b \delta\left(\mathbf{r} - \mathbf{r}^b(t)\right) \mathbf{v}^b(t) \tag{3.25}$$

where $\mathbf{r}^b(t)$, $\mathbf{v}^b(t)$ and $\mathbf{a}^b(t)$ are the particle's position, velocity and acceleration. The ratio $\mu^b \stackrel{def}{=} \alpha^b/\pi^b$ is called the *electric inertial rest mass* of the particle.

62. Of course, (3.23) is equivalent to the standard form of the Lorentz equation:

$$\frac{d}{dt}\left(\frac{\mathbf{v}(t)}{\sqrt{1-\frac{\mathbf{v}(t)^2}{c^2}}}\right) = \pi \left\{\mathbf{E}(\mathbf{r}(t), t) + \mathbf{v}(t) \times \mathbf{B}(\mathbf{r}(t), t)\right\} \tag{3.26}$$

with $\pi = q/m$ in the usual terminology, where q is the passive electric charge and m is the inertial (rest) mass of the particle—that is why we call π *specific* passive electric charge. Nevertheless, it must be clear that for all charged point-particles we introduced *two independent*, empirically meaningful and experimentally testable quantities: specific passive electric charge π and active electric charge α. There is no universal law-like relationship between these two quantities: the ratio between them varies from particle to particle. In the traditional sense, this ratio is, however, nothing but the particle's rest mass.

We must emphasize that the concept of mass so obtained, as defined by only means of electrodynamic quantities, is essentially related to ED, that is to say, to electromagnetic interaction. There seems no way to give a consistent and non-circular operational definition of inertial mass in general, independently of the context of a particular type of physical interaction. Without entering here into the detailed discussion of the problem, we only mention that, for example, Weyl's commonly accepted definition (Jammer 2000, pp. 8–10) and all similar definitions based on the conservation of momentum in particle collisions suffer from the following difficulty. There is no "collision" as a purely "mechanical" process. During a collision the particles are moving in a physical field—or fields—of interaction. Therefore: 1) the system of particles, separately, cannot be regarded as a closed system; 2) the inertial properties of the particles, in fact, reveal themselves in the interactions with the field. Thus, the concepts of inertial rest mass belonging to different interactions differ from each other; whether they are equal (proportional) to each other is a matter of contingent fact of nature.

63. The choice of the *etalon* test particle is, of course, a matter of convention, just as the definitions (D0)–(D4) themselves. It is important to note that all these conventional factors play a constitutive role in the fundamental concepts of electrodynamics (Reichenbach 1965). With these choices we not only make semantic conventions determining the meanings of the terms, but also make a decision about the body of concepts by means of which we grasp physical reality. There are a few things, however, that must be pointed out:

(a) This kind of conventionality does not mean that the physical quantities defined in (D0)–(D4) cannot describe *objective* features of physical reality. It only means that we make a decision which objective features of reality we are dealing with. With another body of conventions we have another body of physical concepts/physical quantities and another body of empirical facts.

(b) On the other hand, it does not mean either that our knowledge of the physical world would not be objective but a product of our conventions. If two theories obtained by starting with two different bodies of conventions are complete enough accounts of the physical phenomena, then they describe the same reality, expressed in terms of different physical quantities. Let us spell out an example: Definition (3.21) is entirely conventional—no objective fact of the world determines the formula on the right hand side. Therefore, we could make another choice, say,

$$\mathbf{j}_{\Theta}\left(\mathbf{r},t\right) \stackrel{def}{=} \Theta^{2}\nabla\times\mathbf{B}\left(\mathbf{r},t\right)-\frac{\partial\mathbf{E}\left(\mathbf{r},t\right)}{\partial t} \tag{3.27}$$

with some $\Theta \neq c$. At first sight, one might think that this choice will alter the speed of electromagnetic waves. This is however not the case. It will be an empirical fact *about* $\mathbf{j}_{\Theta}\left(\mathbf{r},t\right)$ that if a particle b is the only one whose worldline intersects a given space-time region Λ, then for all $(\mathbf{r},t)\in\Lambda$

$$\begin{aligned} \mathbf{j}_{\Theta}\left(\mathbf{r},t\right) &= \alpha^{b}\delta\left(\mathbf{r}-\mathbf{r}^{b}\left(t\right)\right)\mathbf{v}^{b}\left(t\right) \\ &\quad +\left(\Theta^{2}-c^{2}\right)\nabla\times\mathbf{B}(\mathbf{r},t) \end{aligned} \tag{3.28}$$

Now, consider a region where there is no particle. Taking into account (3.28), we have (3.31)–(3.32) and

$$\nabla\cdot\mathbf{E}(\mathbf{r},t) = 0 \tag{3.29}$$

$$\Theta^{2}\nabla\times\mathbf{B}\left(\mathbf{r},t\right)-\frac{\partial\mathbf{E}\left(\mathbf{r},t\right)}{\partial t} = \left(\Theta^{2}-c^{2}\right)\nabla\times\mathbf{B}(\mathbf{r},t) \tag{3.30}$$

which lead to the usual wave equation with propagation speed c. (Of course, in this particular example, one of the possible choices, namely $\Theta = c$, is distinguished by its simplicity. Note, however, that simplicity is not an epistemologically interpretable notion.)

64. All these epistemological issues naturally arise in the context of *empirical verification* whether the relativity principle/covariance is a true law of nature for electrodynamic phenomena. For, empirical verification no doubt requires to know which body of observational data indicates a positive instance of the principle. Without entering here into the discussion of verificationism/operationalism in general, we have only a few short remarks to make.

(a) Our approach is entirely compatible with confirmation/semantic holism. We accept it as true that "our statements about the external world face the tribunal of sense experience not individually but only as a corporate body" (Quine 1951). However, this kind of holism does not deny the empirical content of physical theories, expressible in observational/operational terms. In our view, what semantic holism implies is that the empirical definition of a physical term must not be taken in isolation from the empirical definitions of other terms involved in the definition. Accordingly, we have to formulate our theories on the basis of a sufficiently *large coherent body* of empirical/operational definitions of the fundamental concepts—this is the real holistic approach (super-holistic, if you like). This is why we will formulate electrodynamics in the somewhat unusual terms defined in (D1)–(D4). In this way we obtain a coherent (that is non-circular) body of empirical definitions—avoiding, for example, any reference to the concept of mass.

(b) Our approach relies on the verificationist theory of meaning in the following *very weak* sense: In physics, the meaning of a term standing for a *measurable quantity* which is supposed to characterize an objective feature of physical reality is determined by the empirical operations with which the value of the quantity in question can be ascertained. Such a limited verificationism is widely accepted among physicists; almost all textbooks start with some descriptions of how the basic quantities like electric charge, electric and magnetic field strengths, etc. are *empirically* interpreted. Our concern is that these usual empirical definitions do not satisfy the standard of the above mentioned super-holistic coherence; and the solution of the problem is not entirely trivial.

(c) In any event, the demand for precise operational definitions of electrodynamic quantities emerges not from this epistemological context; not from philosophical ideas about the relationship between physical theories, sense-data, and the external reality; not from the context of questions (Q1) and (Q2). The problem of operational definitions is raised as a problem of pure theoretical physics, in the context of the *inner consistency* of our theories. The reason is that instead of the empirical question (Q2), in fact, we investigate the *theoretical* question (Q4); that requires the theoretical representations of the measurement operations.

3.5 Empirical Facts of Electrodynamics

65. Both "empirical" and "fact" are used in different senses. Statements (E0)–(E4) below are universal generalizations, rather than statements of particular observations. Nevertheless we call them "empirical facts", by which we simply mean that they are truths which can be acquired by *a posteriori* means. Normally, they can be considered as laws obtained by inductive generalization; statements the truths of which can be, in principle, confirmed empirically.

On the other hand, in our context, it is not important how these statements are empirically confirmed. (E0)–(E4) can be regarded as axioms of the Maxwell–Lorentz theory in K. What is important for us is that from these *axioms*, in conjunction with the theoretical representations of the measurement operations, there follow assertions about what the moving observer in K' observes. Section 3.7 will be concerned with these consequences.

(E0) There exist many enough test particles and we can settle them into all required positions and velocities.

Consequently, (D1)–(D4) are sound definitions. From observations about **E**, **B** and the charged point-particles, we have further empirical facts:

(E1) In all situations, the electric and magnetic field strengths satisfy the following two Maxwell equations:

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0 \tag{3.31}$$

$$\nabla \times \mathbf{E}(\mathbf{r},t) + \frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} = 0 \tag{3.32}$$

(E2) Each particle is a charged point-particle, satisfying (D4) with some specific passive electric charge π and active electric charge α. This is also true for the test particles, with—as it follows from the definitions—specific passive electric charge $\pi = 1$.[3]

(E3) If b_1, b_2,..., b_n are the only particles whose worldlines intersect a given space-time region Λ, then for all $(\mathbf{r},t) \in \Lambda$ the source densities are:

$$\varrho(\mathbf{r},t) = \sum_{i=1}^{n} \alpha^{b_i} \delta\left(\mathbf{r} - \mathbf{r}^{b_i}(t)\right) \tag{3.33}$$

$$\mathbf{j}(\mathbf{r},t) = \sum_{i=1}^{n} \alpha^{b_i} \delta\left(\mathbf{r} - \mathbf{r}^{b_i}(t)\right) \mathbf{v}^{b_i}(t) \tag{3.34}$$

Putting facts (E1)–(E3) together, we have the coupled Maxwell–Lorentz equa-

[3] We take it as true that the relativistic Lorentz equation is empirically confirmed. (Cf. Huang 1993)

tions:

$$\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{r},t) &= \sum_{i=1}^{n} \alpha^{b_i}\delta\left(\mathbf{r}-\mathbf{r}^{b_i}(t)\right) && (3.35)\\
c^2\nabla\times\mathbf{B}(\mathbf{r},t)-\frac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t} &= \sum_{i=1}^{n} \alpha^{b_i}\delta\left(\mathbf{r}-\mathbf{r}^{b_i}(t)\right)\mathbf{v}^{b_i}(t) && (3.36)\\
\nabla \cdot \mathbf{B}(\mathbf{r},t) &= 0 && (3.37)\\
\nabla\times\mathbf{E}(\mathbf{r},t)+\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t} &= 0 && (3.38)\\
\frac{\mathbf{a}^{b_i}(t)}{\sqrt{1-\frac{\mathbf{v}^{b_i}(t)^2}{c^2}}} &= \pi^{b_i}\left\{\mathbf{E}\left(\mathbf{r}^{b_i}(t),t\right)+\mathbf{v}^{b_i}(t)\times\mathbf{B}\left(\mathbf{r}^{b_i}(t),t\right)\right.\\
&\quad \left.-\frac{1}{c^2}\mathbf{v}^{b_i}(t)\left(\mathbf{v}^{b_i}(t)\cdot\mathbf{E}\left(\mathbf{r}^{b_i}(t),t\right)\right)\right\} && (3.39)\\
&\qquad (i=1,2,\dots n)
\end{aligned}$$

These are the fundamental equations of ED, describing an interacting system of n particles and the electromagnetic field.

66. Without entering into the details of the problem of classical charged particles (Frisch 2005; Rohrlich 2007; Muller 2007), it must be noted that the Maxwell–Lorentz equations (3.35)–(3.39), exactly in this form, have *no* solution. The reason is the following. In the Lorentz equation of motion (3.23), a small but extended particle can be described with a good approximation by one single specific passive electric charge π^b and one single trajectory $\mathbf{r}^b(t)$. In contrast, however, a similar "idealization" in the source densities (3.24)–(3.25) leads to singularities; the field is singular at precisely the points where the coupling happens: on the trajectory of the particle.

The generally accepted answer to this problem is that (3.24)–(3.25) should not be taken literally. Due to the inner structure of the particle, the real source densities are some "smoothed out" Dirac deltas. Instead of (3.24)–(3.25), therefore, we have some more general equations

$$\begin{aligned}
[\varrho(\mathbf{r},t)] &= \mathcal{R}^b\left[\mathbf{r}^b(t)\right] && (3.40)\\
[\mathbf{j}(\mathbf{r},t)] &= \mathcal{J}^b\left[\mathbf{r}^b(t)\right] && (3.41)
\end{aligned}$$

where $\mathcal{R}^b$ and $\mathcal{J}^b$ are, generally non-linear, operators providing functional relationships between the particle's trajectory $\left[\mathbf{r}^b(t)\right]$ and the source density functions $[\varrho(\mathbf{r},t)]$ and $[\mathbf{j}(\mathbf{r},t)]$. (Notice that (3.24)–(3.25) serve as example of such equations.) The concrete forms of equations (3.40)–(3.41) are determined by the physical laws of the internal world of the particle—which are, supposedly, outside of the scope of classical electrodynamics. At this level of generality, the only thing we can say is that, for a "point-like" (localized) particle, equations (3.40)–(3.41) must be something very close to—but not identical with—equations (3.24)–(3.25). With this explanation, for the sake of simplicity we leave the Dirac deltas in the equations. Also, in some of our statements and calculations the Dirac deltas are essentially used; for example, (E3) and, partly, Theorem 10 and 12 would not be true without the exact point-like source densities (3.24)–(3.25). But a little reflection shows that the statements in question remain approximately true if the particles are approximately point-like, that is, if equations (3.40)–(3.41) are close enough to equations (3.24)–(3.25). To be noted that what is actually essential in (3.24)–(3.25) is not the point-likeness of the particle, but its stability: no matter how the system moves, it remains a localized object.

3.6 Operational Definitions of Electrodynamic Quantities in K'

67. So far we have only considered electrodynamics in a single frame of reference K. Now we turn to the question of how a moving observer describes the same phenomena in K'. The observed phenomena are the same, but the measuring equipments by means of which the phenomena are observed are not entirely the same; instead of being at rest in K, they are co-moving with K'.

Accordingly, we will repeat the operational definitions (D0)–(D4) with the following differences:

1. The "rest test particles" will be at rest relative to reference frame K', that is, *in motion with velocity* $\mathbf{V}$ relative to K.

2. The measuring equipments by means of which the kinematic quantities are ascertained—say, the measuring rods and clocks—will be at rest relative to K', that is, *in motion with velocity* $\mathbf{V}$ relative to K. In other words, the kinematic quantities $\mathbf{r}, t, \mathbf{v}, \mathbf{a}$ in definitions (D0)–(D4) will be *replaced with*—not expressed in terms of— $\mathbf{r}', t', \mathbf{v}', \mathbf{a}'$.

Definition (D0') Particle e is called *(test particle)'* if for all $\mathbf{r}'$ and t'

$$\mathbf{v}'^{e}\left(t'\right)\Big|_{\mathbf{r}'^{e}(t')=\mathbf{r}'} = \mathbf{v}'^{etalon}\left(t'\right)\Big|_{\mathbf{r}'^{etalon}(t')=\mathbf{r}'} \tag{3.42}$$

implies

$$\mathbf{a}'^{e}\left(t'\right)\Big|_{\mathbf{r}'^{e}(t')=\mathbf{r}'} = \mathbf{a}'^{etalon}\left(t'\right)\Big|_{\mathbf{r}'^{etalon}(t')=\mathbf{r}'} \tag{3.43}$$

A (test particle)' e moving with velocity $\mathbf{V}$ relative to K is at rest relative to K', that is, $\mathbf{v}'^{e}=0$. Accordingly:

Definition (D1') *(Electric field strength)'* at point $\mathbf{r}'$ and time t' is defined as the acceleration of an arbitrary (test particle)' e, such that $\mathbf{r}'^{e}(t)=\mathbf{r}'$ and $\mathbf{v}'^{e}(t')=0$:

$$\mathbf{E}'\left(\mathbf{r}',t'\right) \stackrel{def}{=} \mathbf{a}'^{e}(t')\big|_{\mathbf{r}'^{e}(t')=\mathbf{r}';\, \mathbf{v}'^{e}(t')=0} \tag{3.44}$$

Similarly, (magnetic field strength)' is defined by means of how the acceleration $\mathbf{a}'^{e}$ of a rest (test particle)'—rest, of course, relative to K'—changes with a small perturbation of its state of motion, that is, if an infinitesimally small velocity $\mathbf{v}'^{e}$ is imparted to the particle. Just as in (D2), let $\delta' \subset \mathbb{R}^3$ be an arbitrary infinitesimal neighborhood of $0 \in \mathbb{R}^3$. We define the following function:

$$\begin{aligned} \mathbf{U}'^{\mathbf{r}',t'} \quad &: \quad \mathbb{R}^3 \supset \delta' \to \mathbb{R}^3 \\ & \quad \mathbf{U}'^{\mathbf{r}',t'}(\mathbf{v}') \stackrel{def}{=} \mathbf{a}'^{e}(t')\big|_{\mathbf{r}'^{e}(t')=\mathbf{r}';\, \mathbf{v}'^{e}(t')=\mathbf{v}'} \end{aligned} \tag{3.45}$$

Definition (D2') *(Magnetic field strength)'* at point $\mathbf{r}'$ and time t' is

$$\mathbf{B}'(\mathbf{r}',t') \stackrel{def}{=} \begin{pmatrix} \left.\frac{\partial U'^{\mathbf{r}',t'}_y}{\partial v'_z}\right|_{\mathbf{v}'=0} \\ \left.\frac{\partial U'^{\mathbf{r}',t'}_z}{\partial v'_x}\right|_{\mathbf{v}'=0} \\ \left.\frac{\partial U'^{\mathbf{r}',t'}_x}{\partial v'_y}\right|_{\mathbf{v}'=0} \end{pmatrix} \tag{3.46}$$

Definition (D3')

$$\varrho'\left(\mathbf{r}',t'\right) \stackrel{def}{=} \nabla\cdot\mathbf{E}'\left(\mathbf{r}',t'\right) \tag{3.47}$$

$$\mathbf{j}'\left(\mathbf{r}',t'\right) \stackrel{def}{=} c^2\nabla\times\mathbf{B}'\left(\mathbf{r}',t'\right) - \frac{\partial\mathbf{E}'\left(\mathbf{r}',t'\right)}{\partial t'} \tag{3.48}$$

are called *(active electric charge density)'* and *(active electric current density)'*, respectively.

Of course, we have:

Theorem 5.

$$\frac{\partial \varrho'(\mathbf{r}', t')}{\partial t'} + \nabla \cdot \mathbf{j}'(\mathbf{r}', t') = 0 \tag{3.49}$$

Definition (D4') A particle is called *(charged point-particle)'* of *(specific passive electric charge)'* π'^b and of *(active electric charge)'* α'^b if the following is true:

1. It satisfies the relativistic Lorentz equation,

$$\begin{aligned} \frac{\mathbf{a}'^b(t')}{\sqrt{1-\frac{\left(\mathbf{v}'^b(t')\right)^2}{c^2}}} &= \pi'^b \Big\{ \mathbf{E}'\left(\mathbf{r}'^b(t'), t'\right) + \mathbf{v}'^b(t') \times \mathbf{B}'\left(\mathbf{r}'^b(t'), t'\right) \\ &\quad - \frac{1}{c^2}\mathbf{v}'^b(t')\left(\mathbf{v}'^b(t') \cdot \mathbf{E}'\left(\mathbf{r}'^b(t'), t'\right)\right)\Big\} \end{aligned} \tag{3.50}$$

2. If it is the only particle whose worldline intersects a given space-time region Λ', then for all $(\mathbf{r}', t') \in \Lambda'$ the (source densities)' are of the following form:

$$\varrho'(\mathbf{r}', t') = \alpha'^b \delta\left(\mathbf{r}' - \mathbf{r}'^b(t')\right) \tag{3.51}$$

$$\mathbf{j}'(\mathbf{r}', t') = \alpha'^b \delta\left(\mathbf{r}' - \mathbf{r}'^b(t')\right)\mathbf{v}'^b(t') \tag{3.52}$$

where $\mathbf{r}'^b(t')$, $\mathbf{v}'^b(t')$ and $\mathbf{a}'^b(t')$ is the particle's position, velocity and acceleration in K'. The ratio $\mu'^b \stackrel{def}{=} \alpha'^b/\pi'^b$ is called the *(electric inertial rest mass)'* of the particle.

68. It is worthwhile to make a few remarks about some epistemological issues:

(a) The physical quantities defined in (D1)–(D4) *differ* from the physical quantities defined in (D1')–(D4'), simply because the physical situation in which a test particle is at rest relative to K differs from the one in which it is co-moving with K' with velocity $\mathbf{V}$ relative to K; and, as we know *from the laws of electrodynamics in* K, this difference really matters.

Someone might object that if this is so then any two instances of the same measurement must be regarded as measurements of different physical quantities. For, if the difference in the test particle's velocity is enough reason to say that the two operations determine two different quantities, then, by the same token, two operations must be regarded as different operations—and the corresponding quantities as different physical quantities—if the test particle is at different points of space, or the operations simply happen at different moments of time. And this consequence, the objection goes, seems to be absurd: if it were true, then science would not be possible, because we would not have the power to make law-like assertions at all; therefore we must admit that empiricism fails to explain how natural laws are possible, and, as many argue, science cannot do without metaphysical pre-assumptions.

Our response to such an objections is the following. First, concerning the general epistemological issue, we believe, nothing disastrous follows from admitting that two phenomena observed at different place or at different time *are* distinct. And if they are stated as instances of the same phenomenon, this statement is not a logical or metaphysical necessity—derived from some logical/metaphysical pre-assumptions—but an ordinary scientific hypothesis obtained by induction and confirmed or disconfirmed together with the *whole* scientific theory. In fact, this is precisely the case with respect to the definitions of the fundamental electrodynamic quantities. For example, definition (D1) is in fact a family of definitions each belonging to a particular situation individuated by the space-time locus $(\mathbf{r}, t)$.
Second, the question of operational definitions of electrodynamic quantities first of all emerges not from an epistemological context, but from the context of a purely theoretical problem: what do the laws of physics in K say about question (Q2)? In the next section, all the results of the measurement operations defined in (D1')–(D4') will be predicted from the laws of electrodynamics in K. And, electrodynamics itself says that some differences in the conditions are relevant from the point of view of the measured accelerations of the test particles, some others are not; some of the originally distinct quantities are contingently equal, some others not.

(b) From a mathematical point of view, both (D0)–(D4) and (D0')–(D4') are definitions. However, while the choice of the *etalon* test particle and definitions (D0)–(D4) are entirely *conventional*, there is no additional conventionality in (D0')–(D4'). The way in which we define the electrodynamic quantities in inertial frame K' automatically follows from (D0)–(D4) and from the question (Q) we would like to answer; since the question is about the "quantities obtained by the same operational procedures with the same measuring equipments when they are co-moving with K'".

(c) In fact, one of the constituents of the concepts defined in K' is not determined by the operational definitions in K. Namely, the notion of "the same operational procedures with the same measuring equipments when they are co-moving with K'". This is however not an additional freedom of conventionality, but—as we pointed out in Section 2.7—a simple vagueness in our physical theories in K: the vagueness of the general concept of "the same system in the same situation, except that it is, as a whole, in a collective motion with velocity $\mathbf{V}$ relative to K, that is, co-moving with reference frame K'". In any event, in our case, the notion of the only moving measuring device, that is, the notion of "a test particle at rest relative to K'" is quite clear.

3.7 Observations of the Moving Observer

69. Now we have another collection of operationally defined notions, $\mathbf{E}', \mathbf{B}', \varrho', \mathbf{j}'$, the concept of (charged point-particle)' defined in the primed terms, and its properties π', α' and μ'. Normally, one should investigate these quantities experimentally and collect new empirical facts about both the relationships between the primed quantities and about the relationships between the primed quantities and the ones defined in (D1)–(D4). In contrast, we will continue our analysis in another way; following the Lorentzian pedagogy, we will determine from the laws of physics in K what an observer co-moving with K' should observe. In fact, with this method, we will answer our question (Q), on the basis of the laws of electrodynamics in one single frame of reference. We will also see whether the basic equations (3.35)–(3.39) are covariant against these transformations.

Throughout the theorems below, it is important that when we compare, for example, $\mathbf{E}(\mathbf{r}, t)$ with $\mathbf{E}'(\mathbf{r}', t')$, we compare the values of the fields *in one and the same event*, that is, we compare $\mathbf{E}(\mathbf{r}(A), t(A))$ with $\mathbf{E}'(\mathbf{r}'(A), t'(A))$. For the sake of brevity, however, we omit the indication of this fact.

70. The first theorem trivially follows from the fact that the Lorentz transformations of the kinematic quantities are one-to-one:

Theorem 6. *A particle is a (test particle)' if and only if it is a test particle.*

Consequently, we have many enough (test particles)' for definitions (D1')–(D4'); and each is a charged point-particle satisfying the Lorentz equation (3.23) with specific passive electric charge $\pi = 1$.

Theorem 7.

$$E'_x = E_x \tag{3.53}$$
$$E'_y = \frac{E_y - VB_z}{\sqrt{1-\frac{V^2}{c^2}}} \tag{3.54}$$
$$E'_z = \frac{E_z + VB_y}{\sqrt{1-\frac{V^2}{c^2}}} \tag{3.55}$$

Proof. When the (test particle)' is at rest relative to K', it is moving with velocity $\mathbf{v}^e = (V, 0, 0)$ relative to K. From (3.23) (with $\pi = 1$) we have

$$a^e_x = \left(1-\frac{V^2}{c^2}\right)^{\frac{3}{2}} E_x \tag{3.56}$$
$$a^e_y = \sqrt{1-\frac{V^2}{c^2}}\,(E_y - VB_z) \tag{3.57}$$
$$a^e_z = \sqrt{1-\frac{V^2}{c^2}}\,(E_z + VB_y) \tag{3.58}$$

Applying (4.48)–(4.50), we can calculate the acceleration $\mathbf{a}'^e$ in K', and, accordingly, we find

$$E'_x = a'^e_x = \frac{a^e_x}{\left(1-\frac{V^2}{c^2}\right)^{\frac{3}{2}}} = E_x \tag{3.59}$$
$$E'_y = a'^e_y = \frac{a^e_y}{1-\frac{V^2}{c^2}} = \frac{E_y - VB_z}{\sqrt{1-\frac{V^2}{c^2}}} \tag{3.60}$$
$$E'_z = a'^e_z = \frac{a^e_z}{1-\frac{V^2}{c^2}} = \frac{E_z + VB_y}{\sqrt{1-\frac{V^2}{c^2}}} \tag{3.61}$$

□

Theorem 8.

$$B'_x = B_x \tag{3.62}$$
$$B'_y = \frac{B_y + \frac{V}{c^2}E_z}{\sqrt{1-\frac{V^2}{c^2}}} \tag{3.63}$$
$$B'_z = \frac{B_z - \frac{V}{c^2}E_y}{\sqrt{1-\frac{V^2}{c^2}}} \tag{3.64}$$

Proof. Consider for instance B'_x. By definition,

$$B'_x = \left.\frac{\partial {U'}_y^{\mathbf{r}',t'}}{\partial {v'}_z}\right|_{\mathbf{v}'=0} \tag{3.65}$$

According to (3.45), the value of ${U'}_y^{\mathbf{r}',t'}(\mathbf{v}')$ is equal to

$$\left.{a'}^e_y\right|_{\mathbf{r}'^e(t')=\mathbf{r}';\,\mathbf{v}'^e(t')=\mathbf{v}'} \tag{3.66}$$

that is, the y-component of the acceleration of a (test particle)' e in a situation in which $\mathbf{r}'^e(t') = \mathbf{r}'$ and $\mathbf{v}'^e(t') = \mathbf{v}'$. Accordingly, in order to determine B'_x (3.65) we have to determine

$$\left.\frac{d}{dw}\right|_{w=0}\left(\left.{a'}^e_y\right|_{\mathbf{r}'^e(t')=\mathbf{r}';\,\mathbf{v}'^e(t')=(0,0,w)}\right) \tag{3.67}$$

Now, according to (4.47), condition $\mathbf{v}'^e = (0,0,w)$ corresponds to

$$\mathbf{v}^e = \left(V, 0, w\sqrt{1-\frac{V^2}{c^2}}\right) \tag{3.68}$$

Substituting this velocity into (3.23), we have:

$$a^e_y = \sqrt{1-\frac{V^2+w^2\left(1-\frac{V^2}{c^2}\right)}{c^2}}\left(E_y + w\sqrt{1-\frac{V^2}{c^2}}B_x - VB_z\right) \tag{3.69}$$

Applying (4.51), one finds:

$$\begin{aligned}
{a'}^e_y &= \frac{a^e_y}{1-\frac{V^2}{c^2}} = \frac{\sqrt{1-\frac{V^2+w^2\left(1-\frac{V^2}{c^2}\right)}{c^2}}}{1-\frac{V^2}{c^2}}\left(E_y + w\sqrt{1-\frac{V^2}{c^2}}B_x - VB_z\right) \\
&= \sqrt{\frac{1-\frac{w^2}{c^2}}{1-\frac{V^2}{c^2}}}\left(E_y + w\sqrt{1-\frac{V^2}{c^2}}B_x - VB_z\right)
\end{aligned} \tag{3.70}$$

Differentiating with respect to w at $w = 0$, we obtain

$$B'_x = B_x \tag{3.71}$$

The other components can be obtained in the same way. □

Theorem 9.

$$\varrho' = \frac{\varrho - \frac{V}{c^2} j_x}{\sqrt{1-\frac{V^2}{c^2}}} \tag{3.72}$$

$$j'_x = \frac{j_x - V\varrho}{\sqrt{1-\frac{V^2}{c^2}}} \tag{3.73}$$

$$j'_y = j_y \tag{3.74}$$

$$j'_z = j_z \tag{3.75}$$

Proof. In (3.47) and (3.48), substituting $\mathbf{E}'$ and $\mathbf{B}'$ with the right-hand-sides of (3.53)–(3.55) and (3.62)–(3.64), $\mathbf{r}$ and t with the inverse of (4.39)–(4.42), then differentiating the composite function and taking into account (3.20)–(3.21), we get (3.72)–(3.75). □

Theorem 10. *A particle b is charged point-particle of specific passive electric charge π^b and of active electric charge α^b if and only if it is a (charged point-particle)' of (specific passive electric charge)' π'^b and of (active electric charge)' α'^b, such that $\pi'^b = \pi^b$ and $\alpha'^b = \alpha^b$.*

Proof. First we prove (3.50). For the sake of simplicity, we will verify this in case of $\mathbf{v}'^b = (0,0,w)$. We can use (3.69):

$$a^b_y = \pi^b \sqrt{1 - \frac{V^2 + w^2\left(1-\frac{V^2}{c^2}\right)}{c^2}} \left(E_y + w\sqrt{1-\frac{V^2}{c^2}} B_x - V B_z\right) \tag{3.76}$$

From (4.51), (3.54), (3.62), and (3.64) we have

$$\begin{aligned} a'^b_y &= \pi^b \sqrt{1-\frac{w^2}{c^2}} \left(E'_y + wB'_x\right) \\ &= \left.\left\{\pi^b \sqrt{1-\frac{\left(\mathbf{v}'^b\right)^2}{c^2}} \left(\mathbf{E}' - \frac{1}{c^2}\mathbf{v}'^b \left(\mathbf{v}'^b \cdot \mathbf{E}'\right) + \mathbf{v}'^b \times \mathbf{B}'\right)\right\}_y \right|_{\mathbf{v}'^b = (0,0,w)} \end{aligned} \tag{3.77}$$

Similarly,

$$
\begin{aligned}
a_x'^b &= \pi^b\sqrt{1-\frac{w^2}{c^2}}\left(E_x'-wB_y'\right) \\
&= \left\{\pi^b\sqrt{1-\frac{\left(\mathbf{v}'^b\right)^2}{c^2}}\left(\mathbf{E}'-\frac{1}{c^2}\mathbf{v}'^b\left(\mathbf{v}'^b\cdot\mathbf{E}'\right)+\mathbf{v}'^b\times\mathbf{B}'\right)\right\}_x\Bigg|_{\mathbf{v}'^b=(0,0,w)} \qquad (3.78)\\
a_z'^b &= \pi^b\left(1-\frac{w^2}{c^2}\right)^{\frac{3}{2}}E_z' \\
&= \left\{\pi^b\sqrt{1-\frac{\left(\mathbf{v}'^b\right)^2}{c^2}}\left(\mathbf{E}'-\frac{1}{c^2}\mathbf{v}'^b\left(\mathbf{v}'^b\cdot\mathbf{E}'\right)+\mathbf{v}'^b\times\mathbf{B}'\right)\right\}_z\Bigg|_{\mathbf{v}'^b=(0,0,w)} \qquad (3.79)
\end{aligned}
$$

That is, (3.50) is satisfied, indeed.

In the second part, we show that (3.51)–(3.52) are nothing but (3.24)–(3.25) expressed in terms of $\mathbf{r}', t', \varrho'$ and $\mathbf{j}'$, with $\alpha'^b = \alpha^b$.

It will be demonstrated for a particle of trajectory $\mathbf{r}'^b(t') = (wt', 0, 0)$. Applying (4.46), (3.24)–(3.25) have the following forms:

$$
\begin{aligned}
\varrho(\mathbf{r},t) &= \alpha^b\delta(x-\beta t)\,\delta(y)\,\delta(z) \qquad (3.80)\\
\mathbf{j}(\mathbf{r},t) &= \alpha^b\delta(x-\beta t)\,\delta(y)\,\delta(z)\begin{pmatrix}\beta\\0\\0\end{pmatrix} \qquad (3.81)
\end{aligned}
$$

where $\beta = \frac{w+V}{1+(wV/c^2)}$. $\mathbf{r}, t, \varrho$ and $\mathbf{j}$ can be expressed with the primed quantities by applying the inverse of (4.39)–(4.42) and (3.72)–(3.75):

$$
\begin{aligned}
\frac{\varrho'(\mathbf{r}',t')+\frac{V}{c^2}j'_x(\mathbf{r}',t')}{\sqrt{1-\frac{V^2}{c^2}}} &= \alpha^b\delta\left(\frac{x'+Vt'}{\sqrt{1-\frac{V^2}{c^2}}}-\beta\frac{t'+\frac{V}{c^2}x'}{\sqrt{1-\frac{V^2}{c^2}}}\right)\delta(y')\,\delta(z') \qquad (3.82)\\
\frac{j'_x(\mathbf{r}',t')+V\varrho'(\mathbf{r}',t')}{\sqrt{1-\frac{V^2}{c^2}}} &= \alpha^b\delta\left(\frac{x'+Vt'}{\sqrt{1-\frac{V^2}{c^2}}}-\beta\frac{t'+\frac{V}{c^2}x'}{\sqrt{1-\frac{V^2}{c^2}}}\right)\\
&\quad\times\delta(y')\,\delta(z')\,\beta \qquad (3.83)\\
j'_y(\mathbf{r}',t') &= 0 \qquad (3.84)\\
j'_z(\mathbf{r}',t') &= 0 \qquad (3.85)
\end{aligned}
$$

One can solve this system of equations for ϱ' and j'_x:

$$\begin{aligned}
\varrho'(\mathbf{r}', t') &= \alpha^b \delta(x' - wt')\,\delta(y')\,\delta(z') && (3.86)\\
\mathbf{j}'(\mathbf{r}', t') &= \alpha^b \delta(x' - wt')\,\delta(y')\,\delta(z') \begin{pmatrix} w \\ 0 \\ 0 \end{pmatrix} && (3.87)
\end{aligned}$$

□

Theorem 11.

$$\begin{aligned}
\nabla \cdot \mathbf{B}'(\mathbf{r}', t') &= 0 && (3.88)\\
\nabla \times \mathbf{E}'(\mathbf{r}', t') + \frac{\partial \mathbf{B}'(\mathbf{r}', t')}{\partial t'} &= 0 && (3.89)
\end{aligned}$$

Proof. Expressing (3.31)–(3.32) in terms of $\mathbf{r}', t', \mathbf{E}'$ and $\mathbf{B}'$ by means of (4.39)–(4.42), (3.53)–(3.55) and (3.62)–(3.64), we have

$$\begin{aligned}
\nabla \cdot \mathbf{B}' - \frac{V}{c^2}\left(\nabla \times \mathbf{E}' + \frac{\partial \mathbf{B}'}{\partial t'}\right)_x &= 0 && (3.90)\\
\left(\nabla \times \mathbf{E}' + \frac{\partial \mathbf{B}'}{\partial t'}\right)_x - V\nabla \cdot \mathbf{B}' &= 0 && (3.91)\\
\left(\nabla \times \mathbf{E}' + \frac{\partial \mathbf{B}'}{\partial t'}\right)_y &= 0 && (3.92)\\
\left(\nabla \times \mathbf{E}' + \frac{\partial \mathbf{B}'}{\partial t'}\right)_z &= 0 && (3.93)
\end{aligned}$$

which is equivalent to (3.88)–(3.89). □

Theorem 12. *If b_1, b_2,..., b_n are the only particles whose worldlines intersect a given space-time region Λ', then for all $(\mathbf{r}', t') \in \Lambda'$ the (source densities)' are:*

$$\begin{aligned}
\varrho'(\mathbf{r}', t') &= \sum_{i=1}^{n} \alpha^{b_i} \delta\left(\mathbf{r}' - \mathbf{r}'^{b_i}(t')\right) && (3.94)\\
\mathbf{j}'(\mathbf{r}', t') &= \sum_{i=1}^{n} \alpha^{b_i} \delta\left(\mathbf{r}' - \mathbf{r}'^{b_i}(t')\right) \mathbf{v}'^{b_i}(t') && (3.95)
\end{aligned}$$

Proof. Due to Theorem 10, each (charged point-particle)' is a charged point-particle with $\alpha'^b = \alpha^b$. Therefore, we only need to prove that equations (3.94)–(3.95) amount to (3.33)–(3.34) expressed in the primed variables. On the left hand side of (3.33)–(3.34), ϱ and $\mathbf{j}$ can be expressed by means of (3.72)–(3.75); on the right hand side, we take $\alpha'^b = \alpha^b$, and apply the inverse of (4.39)–(4.42), just as in the derivation of

(3.86)–(3.87). From the above, we obtain:

$$\begin{aligned}
\varrho'\left(\mathbf{r}',t'\right)+\frac{V}{c^2}j'_x\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right) \\
&\quad +\frac{V}{c^2}\sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)v'^{b_i}_x\left(t'\right) && (3.96)\\
j'_x\left(\mathbf{r}',t'\right)+V\varrho'\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)v'^{b_i}_x\left(t'\right) \\
&\quad +V\sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right) && (3.97)\\
j'_y\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)v'^{b_i}_y\left(t'\right) && (3.98)\\
j'_z\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)v'^{b_i}_z\left(t'\right) && (3.99)
\end{aligned}$$

Solving these linear equations for ϱ' and $\mathbf{j}'$ we obtain (3.94)–(3.95). □

Combining all the results we obtained in Theorems 10–12, we have

$$\begin{aligned}
\nabla\cdot\mathbf{E}'\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha'^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right) && (3.100)\\
c^2\nabla\times\mathbf{B}'\left(\mathbf{r}',t'\right)-\frac{\partial\mathbf{E}'\left(\mathbf{r}',t'\right)}{\partial t'} &= \sum_{i=1}^{n}\alpha'^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)\mathbf{v}'^{b_i}\left(t'\right) && (3.101)\\
\nabla\cdot\mathbf{B}'\left(\mathbf{r}',t'\right) &= 0 && (3.102)\\
\nabla\times\mathbf{E}'\left(\mathbf{r}',t'\right)+\frac{\partial\mathbf{B}'\left(\mathbf{r}',t'\right)}{\partial t'} &= 0 && (3.103)\\
\frac{\mathbf{a}'^{b_i}\left(t'\right)}{\sqrt{1-\frac{\mathbf{v}'^{b_i}\left(t'\right)^2}{c^2}}} &= \pi'^{b_i}\left\{\mathbf{E}'\left(\mathbf{r}'^{b_i}\left(t'\right),t'\right)\right. \\
&\quad +\mathbf{v}'^{b_i}\left(t'\right)\times\mathbf{B}'\left(\mathbf{r}'^{b_i}\left(t'\right),t'\right) \\
&\quad \left.-\mathbf{v}'^{b_i}\left(t'\right)\frac{\mathbf{v}'^{b_i}\left(t'\right)\cdot\mathbf{E}'\left(\mathbf{r}'^{b_i}\left(t'\right),t'\right)}{c^2}\right\} && (3.104)\\
&\qquad (i=1,2,\ldots n)
\end{aligned}$$

3.8 Are the Transformation Rules Derived from Covariance True?

71. Our main concern in this chapter was: On what grounds can the transformation rules for the electrodynamic quantities derived from the assumption of covariance—

hence the hypothesis of covariance itself—be considered as verified facts of the physical world? Now everything is at hand to declare that the these transformation rules are in fact true, at least in the sense that they are derivable from the laws of electrodynamics in a single frame of reference—*without the prior assumption of covariance*. For, Theorems 7 and 8 show the well-known transformation rules for the field variables. What Theorem 9 asserts is nothing but the well-known transformation rule for charge density and current density. Finally, Theorem 10 shows that a particle's electric specific passive charge, active charge and electric rest mass are invariant Lorentz scalars. And, of course, these results make it possible to use the well-known covariant formulation of electrodynamics.

At this point, having ascertained the transformation rules, we can recognize that equations (3.100)–(3.104) are nothing but equations (3.35)–(3.39) expressed in the primed variables. At the same time, (3.100)–(3.104) are manifestly of the same form as (3.35)–(3.39). Therefore, we proved that the Maxwell–Lorentz equations are *indeed* covariant against the *real* transformations of the kinematic and electrodynamic quantities. In fact, we proved more:

- The Lorentz equation of motion (3.39) is covariant separately.
- The four Maxwell equations (3.35)–(3.38) constitute a covariant set of equations, separately from (3.39).
- (3.35)–(3.36) constitute a covariant set of equations, separately.
- (3.37)–(3.38) constitute a covariant set of equations, separately.

None of these statements follows automatically from the fact that (3.35)–(3.39) form a covariant system of equations (cf. Paragraph **43**).

72. It is of interest to notice that all these results hinge on the *relativistic* version of the Lorentz equation, in particular, on the "relativistic mass-formula". Without factor $\left(1-\left(\mathbf{v}^b\right)^2/c^2\right)^{-\frac{1}{2}}$ in (3.39), the proper transformation rules were different and the Maxwell equations were not covariant—against the proper transformations.

Chapter 4

Does the Principle of Relativity Hold in Electrodynamics?

4.1 The Problem of M

73. Assume that the physical equations in question are covariant, that is (2.85) is satisfied. As we have pointed out in Chapter 2, the principle of relativity requires more than this. The covariance of the equations guarantees that the "Lorentz boosted behavior" $\Lambda^{-1}(P(F))$ is *a* solution of the equations whenever F is one. The relativity principle *additionally* requires this solution to be identical with $M(F)$, the solution describing the same phenomenon as F, stipulated to describe the physical system in question at rest, but when the system is, as a whole, in motion with an additional collective velocity.

One might think that to ascertain whether the relativity principle holds for the system in question we simply need to check this identity; or one of its variants (2.102) or (2.108) expressed in terms of the corresponding initial and boundary conditions. But, when can we say that a solution $M(F)$ corresponds to the "same phenomenon as F, stipulated to describe the physical system at rest, but when the system is, as a whole, in motion with an additional collective velocity"? In principle, an arbitrary solution F can be regarded as the description of the "system at rest" (as long as it satisfies the minimum requirement (M) introduced in Paragraph **46**). The question is: for an arbitrary F what is the corresponding $M(F)$? Without answering this question the statement of the principle is simply meaningless.

As the spring example in Paragraph **34** illustrates, in classical mechanics the concept of M has a straightforward understanding; moreover, with this choice of M the relativity principle is satisfied. However, one can show simple examples of relativistic systems in which the meaning of M is vague; and with reasonable choice of M the relativity principle is violated (Szabó 2004). In this chapter we shall see that in electrodynamics the situation is in some sense even worse.

Recall what we called 'minimum requirement for the relativity principle' (M) in Paragraph **46**: whatever is the definition of $M : \mathcal{E} \to \mathcal{E}$, a minimal requirement for it to have the assumed physical meaning, in other words, a minimal requirement for the relativity principle to be a meaningful statement is that relations $F \in \mathcal{E}$ must describe a phenomenon which can be meaningfully characterized as such that the physical system exhibiting this phenomenon is co-moving with some inertial frame of reference, that is, it is at rest or in motion with some velocity relative to any given inertial frame. It will be argued that even this minimum requirement is failed to be satisfied by a general electrodynamic system.

Nevertheless, we would like to raise this problem in the context of a more general metaphysical issue. For, as we will now see, the problem of motion is deeply intertwined with the more fundamental problem of persistence.

4.2 Motion and Persistence

74. There is a long debate in contemporary metaphysics whether and in what sense instantaneous velocity can be regarded as an intrinsic property of an object at a given moment of time (Butterfield 2006; Arntzenius 2000; Tooley 1988; Hawley 2001, 76–80; Sider 2001, 34–35). What is relevant from this debate to our present concern is—in which there seems to be a consensus—that "the notion of velocity presupposes the persistence of the object concerned" (Butterfield 2005, 257). In fact, as we will see, velocity occurs as a feature of the way in which the object persists. Therefore, the concept that an object as a whole is in motion with some velocity can be expressed in the same terms as the persistence of the object.

75. It is common to all theories of persistence—endurantism vs. perdurantism—that a persisting entity needs to have some package of individuating properties, in terms of which one can express that two things in two different spatiotemporal regions are identical, or at least constitute different temporal parts of the same entity. Butterfield writes:

> I believe that [the criteria of identity] are largely independent of the endurantism–perdurantism debate; and in particular, that endurantism and perdurantism [...] face some common questions about criteria of identity, and can often give the same, or similar, answers to them. [...]

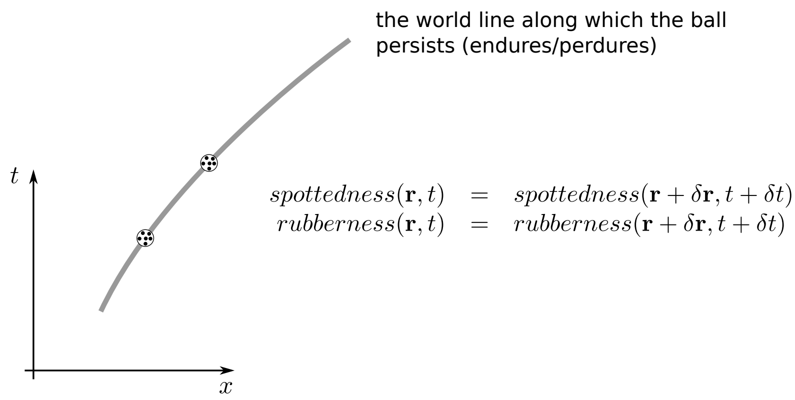

Figure 4.1: A small, "point-like" ball can be individuated by its spottednes, rubberness, etc.

> [A]ll parties need to provide criteria of identity for objects, presumably invoking the usual notions of qualitative similarity and-or causation (Butterfield 2005, 248–289)

Without loss of generality we may assume that each of these individuating properties can be characterized as such that a certain (real valued) quantity f_i takes a certain value; more precisely, the spatiotemporal distribution of this quantity, $f_i(\mathbf{r}, t)$, takes a certain local value at a spatiotemporal locus. Accordingly, we express persistence—and, in this way, the concept of velocity—in terms of these distributions of individuating quantities. We proceed in three heuristic steps.

I.

First we consider the persistence of a point-like physical object. The fact that a point-like object (or its temporal part), occupying a small place at point $\mathbf{r}$ in space at the moment of time t, instantiates a certain property in that moment can be expressed by the fact that the corresponding quantity $f_i(\mathbf{r}, t)$ takes a certain local value. For example: the ball in Fig. 4.1 can be described by the spatial distributions of two quantities, $spottedness(\mathbf{r}, t)$ and $rubberness(\mathbf{r}, t)$—taking value, say, 1 where spottedness/rubberness is instantiated and 0 otherwise.

To express the fact of persistence we will use the distributions of a given package of individuating quantities $\{f_i(\mathbf{r}, t)\}_{i=1}^{n}$. Different theories may disagree in the actual content of this package, of course. We only assume that these quantities together are capable to express the fact that two things in two different space-time points are identical (endurance), or, constitute different temporal parts of the same entity (perdurance); in other words, they trace out the world-line along which the point-like object persists. Expressing identity in terms of equality of the individuating quantities in the different spatiotemporal regions, we have

$$f_i(\mathbf{r}, t) = f_i(\mathbf{r}', t') \qquad (i = 1, 2, \ldots, n) \tag{4.1}$$

for any two points $(\mathbf{r}, t)$ and $(\mathbf{r}', t')$ along the world-line (Fig. 4.1). Introducing the *average velocity* as $\mathbf{v} = \frac{\mathbf{r}'-\mathbf{r}}{t'-t}$, we can write:

$$f_i(\mathbf{r}, t) = f_i(\mathbf{r} + \mathbf{v}\delta t, t + \delta t) \qquad (i = 1, 2, \ldots, n) \tag{4.2}$$

Assume that all functions in $\{f_i(\mathbf{r}, t)\}_{i=1}^{n}$ are smooth (if not, they can be approximated as closely as required for physics by smooth functions). Taking (4.2) for a small, infinitesimal interval of time, and expressing it in a differential form, we have

$$-\frac{\partial f_i(\mathbf{r}, t)}{\partial t} = \nabla f_i(\mathbf{r}, t) \cdot \mathbf{v}(t) \qquad (i = 1, 2, \ldots, n) \tag{4.3}$$

where $\mathbf{v}(t)$ is the instantaneous velocity. In components:

$$-\frac{\partial f_i(\mathbf{r}, t)}{\partial t} = v_x \frac{\partial f_i(\mathbf{r}, t)}{\partial x} + v_y \frac{\partial f_i(\mathbf{r}, t)}{\partial y} + v_z \frac{\partial f_i(\mathbf{r}, t)}{\partial z} \qquad (i = 1, 2, \ldots, n) \tag{4.4}$$

Of course, the concrete world-line along which the object persists may be varied. Thus, equations (4.3) with some instantaneous velocity constitute a *necessary* condition the individuating quantities must satisfy in every space-time point where the object persists. Let us call them the *equations of point-like persistence.*

II.

Now we make a straightforward extension of the above results to the case of an extended object. Assume that the fine-grained structure of an extended object also can be described in terms of the distributions of some, probably more fundamental, quantities (Fig. 4.2). And, therefore, the identity of the persisting object can be expressed in terms of an individuating package of these distributions, $\{f_i(\mathbf{r}, t)\}_{i=1}^{n}$. It is a straightforward generalization of the idea expressed in equation (4.3) to say

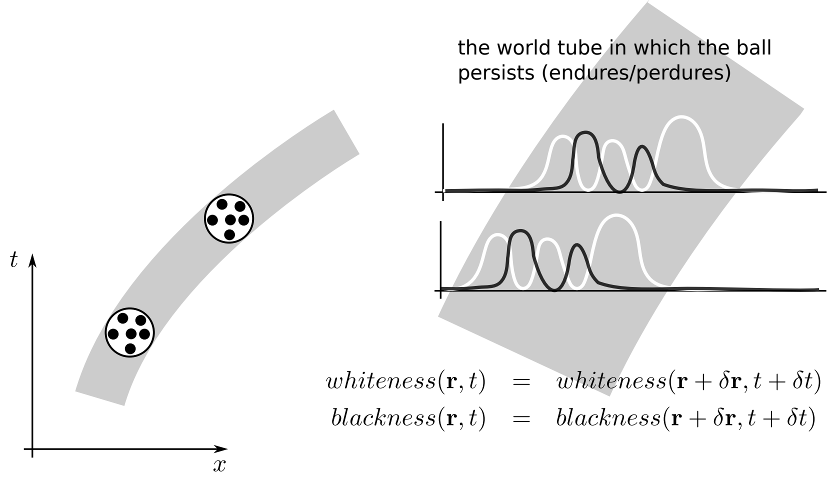

Figure 4.2: A spotted ball, as extended object, can be characterized by the distributions of whiteness and blackness

that an extended object persists if there is a velocity vector $\mathbf{v}(t)$ for every moment of time, such that equation (4.3) is satisfied in all space-time points $(\mathbf{r}, t)$ belonging to the space-time tube swept by the extended object.

However, this describes only a particular situation when the extended object persists like a rigid body in translational motion. The instantaneous velocity $\mathbf{v}(t)$ is the same everywhere in the spatial region occupied by the object. Consequently, the spatial distributions $f_i(\mathbf{r}, t = const)$ are simply translating with a universal velocity, without deformation. Of course, generally this is not necessarily the case. For example, the ball in Fig. 4.3 preserves its identity even though it rotates and inflates.

III.

Concerning the general case, imagine an extended object with a more complex behavior. Let Σ_t and $\Sigma_{t+\delta t}$ denote the spatial regions occupied by the object at time t and $t + \delta t$. The object can change in various sense. Even if $\Sigma_t = \Sigma_{t+\delta t}$, the spatial distributions of its local properties may change, in the sense that for several distributions $f_i(\mathbf{r}, t) \not\equiv f_i(\mathbf{r}, t + \delta t)$. Moreover, Σ_t and $\Sigma_{t+\delta t}$ may differ not only in their location but also in size and shape. All these changes manifest themselves in the spatiotemporal distributions of local properties, that is, in the distributions $f_i(\mathbf{r}, t)$. For example, all changes, the translation, the rotation, and the inflation of the ball in Fig. 4.3 are expressible in terms of the distributions like $whiteness(\mathbf{r}, t)$ and $blackness(\mathbf{r}, t)$.

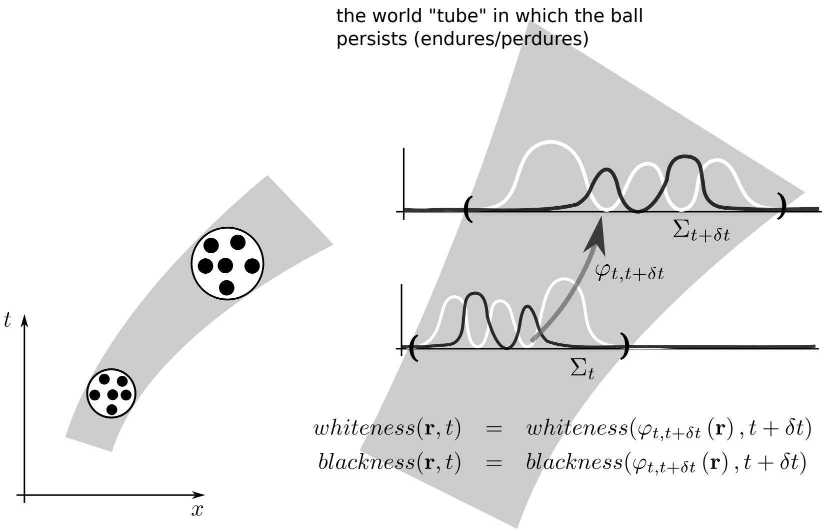

Figure 4.3: the ball preserves its identity even though it may rotate or inflate

Now, how can we describe the persistence of such an object? What conditions the distributions $f_i(\mathbf{r},t)$ must satisfy in order to count the two things in Σ_t and $\Sigma_{t+\delta t}$ as identical, or as two temporal parts of the same object? The conditions we are looking for have to express some similarities between the values of $f_i(\mathbf{r},t)$ in Σ_t and the values of $f_i(\mathbf{r},t+\delta t)$ in $\Sigma_{t+\delta t}$. On the basis of our previous considerations in points I and II and the examples like the inflating-rotating ball, we claim that the general form of such conditions are the following. There must exist a package of relevant, individuating distributions $\{f_i(\mathbf{r},t)\}_{i=1}^n$ and a mapping $\varphi_{t,t+\delta t}:\Sigma_t\to\Sigma_{t+\delta t}$, such that

$$f_i(\mathbf{r},t) = f_i(\varphi_{t,t+\delta t}(\mathbf{r}), t+\delta t) \qquad (i=1,2,\ldots,n) \tag{4.5}$$

Notice that the only non-trivial requirement concerning $\varphi_{t,t+\delta t}$ is that it must be common for all individuating distributions $f_i(\mathbf{r},t)$. Intuitively this means that if a local part of the objects at $\mathbf{r}$ instantiates some local individuating properties then its counterpart at point $\varphi_{t,t+\delta t}(\mathbf{r})$ instantiates the same local individuating properties. But this fact by no means implies that the extended object necessarily consists of atomic entities—pointlike or non-pointlike—persisting in the sense of points I or II. Just the contrary, the general notion of persistence defined by (4.5) satisfies a kind of downward mereological principle: if the whole extended object persists in the sense of (4.5) then all (arbitrarily small) local parts of the object persist in the same sense.

Assuming that $\varphi_{t,t+\delta t}(\mathbf{r})$ is smooth and $\varphi_{t,t} = id_{\Sigma_t}$, one can express (4.5) in the following differential form:

$$-\frac{\partial f_i(\mathbf{r},t)}{\partial t} = \nabla f_i(\mathbf{r},t) \cdot \mathbf{v}(\mathbf{r},t) \qquad (i = 1, 2, \ldots, n) \tag{4.6}$$

where

$$\mathbf{v}(\mathbf{r},t) = \left. \frac{\partial \varphi_{t,t'}(\mathbf{r})}{\partial t'} \right|_{t'=t}$$

$\mathbf{v}(\mathbf{r},t)$ can be interpreted as the instantaneous velocity field characterizing the motion of the local part of the extended entity at the spatiotemporal locus $(\mathbf{r},t)$.

Taking into account that the concrete mapping $\varphi_{t,t+\delta t} : \Sigma_t \to \Sigma_{t+\delta t}$ may be varied, equations (4.6) with some suitable instantaneous velocity field $\mathbf{v}(\mathbf{r},t)$ constitute a *necessary* condition the distributions of individuating quantities must satisfy in every space-time point where the extended object persists. Let us call them the *equations of persistence*.

76. In conclusion, we have seen that:

- The relativity principle requires the notion that a physical object, as a whole, is in motion with some (zero or nonzero) velocity relative to an inertial frame of reference.
- This notion presupposes the persistence of the object.
- Persistence requires the existence of a field of local instantaneous velocity, with which the distributions of the individuating quantities satisfy the equations of persistence.

4.3 The Case of a General Electrodynamic System

77. We will now deal with an electrodynamic system, that is a coupled system of charged particles and the electromagnetic field. The system is described by the Maxwell–Lorentz equations (3.35)–(3.39).

Let us first take the well-known textbook example: the static versus uniformly moving 'charged particle + the coupled electromagnetic field' system. As was demonstrated in Paragraph **33**, this system nicely satisfies the relativity principle.

In this example, it is easy to verify that (2.77)—as a particular case, (2.76) too—satisfies the equations of persistence (4.3) with constant and homogeneous velocity field $\mathbf{V} = (V, 0, 0)$,[1] in the following sense:[2]

$$-\frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} = \mathsf{D}\mathbf{E}(\mathbf{r},t)\mathbf{V} \tag{4.7}$$

$$-\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} = \mathsf{D}\mathbf{B}(\mathbf{r},t)\mathbf{V} \tag{4.8}$$

$$-\frac{\partial \varrho(\mathbf{r},t)}{\partial t} = \nabla\varrho(\mathbf{r},t)\cdot\mathbf{V} \tag{4.9}$$

where $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are regarded as intrinsic properties of the electromagnetic field, the components of which belong to the package of individuating properties. Or, in the more expressive form of (4.1),

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r}+\mathbf{V}\delta t, t+\delta t) \tag{4.10}$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}(\mathbf{r}+\mathbf{V}\delta t, t+\delta t) \tag{4.11}$$

$$\varrho(\mathbf{r},t) = \varrho(\mathbf{r}+\mathbf{V}\delta t, t+\delta t) \tag{4.12}$$

This picture is in complete accordance with the standard realistic interpretation of

[1] It must be pointed out that velocity $\mathbf{V}$ conceptually differs from the speed of light c. Basically, c is a constant of nature in the Maxwell–Lorentz equations, which can emerge in the solutions of the equations; and, in some cases, it can be interpreted as the velocity of propagation of changes in the electromagnetic field. For example, in our case, the stationary field of a uniformly moving point charge, in collective motion with velocity $\mathbf{V}$, can be constructed from the superposition of retarded potentials, in which the retardation is calculated with velocity c; nevertheless, the two velocities are different concepts. To illustrate the difference, consider the fields of a charge at rest (2.76), and in motion (2.77). The speed of light c plays the same role in both cases. Both fields can be constructed from the superposition of retarded potentials in which the retardation is calculated with velocity c. Also, in both cases, a small local perturbation in the field configuration would propagate with velocity c. But still, there is a consensus to say that the system described by (2.76) is at rest while the one described by (2.77) is moving with velocity $\mathbf{V}$ (together with K', relative to K.) A good analogy would be a Lorentz contracted moving rod: $\mathbf{V}$ is the velocity of the rod, which differs from the speed of sound in the rod.

[2] In $\mathsf{D}\mathbf{E}(\mathbf{r},t)$ and $\mathsf{D}\mathbf{B}(\mathbf{r},t)$, D denotes the spatial derivative operator (Jacobian for variables x, y and z). That is, in components we have:

$$\begin{aligned}
-\frac{\partial E_x(\mathbf{r},t)}{\partial t} &== V_x\frac{\partial E_x(\mathbf{r},t)}{\partial x} + V_y\frac{\partial E_x(\mathbf{r},t)}{\partial y} + V_z\frac{\partial E_x(\mathbf{r},t)}{\partial z}\\
-\frac{\partial E_y(\mathbf{r},t)}{\partial t} &= V_x\frac{\partial E_y(\mathbf{r},t)}{\partial x} + V_y\frac{\partial E_y(\mathbf{r},t)}{\partial y} + V_z\frac{\partial E_y(\mathbf{r},t)}{\partial z}\\
&\vdots\\
-\frac{\partial B_z(\mathbf{r},t)}{\partial t} &= V_x\frac{\partial B_z(\mathbf{r},t)}{\partial x} + V_y\frac{\partial B_z(\mathbf{r},t)}{\partial y} + V_z\frac{\partial B_z(\mathbf{r},t)}{\partial z}\\
-\frac{\partial \varrho(\mathbf{r},t)}{\partial t} &= V_x\frac{\partial \varrho(\mathbf{r},t)}{\partial x} + V_y\frac{\partial \varrho(\mathbf{r},t)}{\partial y} + V_z\frac{\partial \varrho(\mathbf{r},t)}{\partial z}
\end{aligned}$$

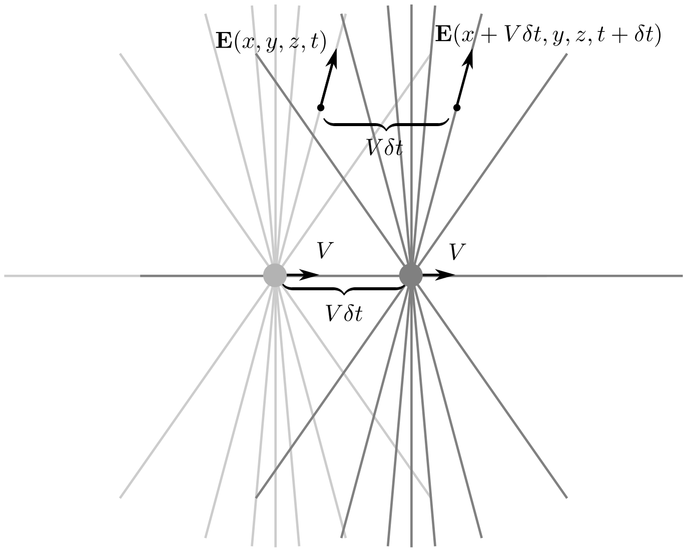

Figure 4.4: The stationary field of a uniformly moving point charge is in collective motion together with the point charge

electromagnetic field:

> In the standard interpretation of the formalism, the field strengths **B** and **E** are interpreted realistically: The interaction between charged particles are mediated by the electromagnetic field, which is ontologically on a par with charged particles and the state of which is given by the values of the field strengths. (Frisch 2005, 28)

Thus, in this particular example the necessary conditions of the persistence of the 'particle + electromagnetic field' system are clearly satisfied; therefore we may have, and actually have, a meaningful concept of 'the particle + electromagnetic field' system co-moving with an inertial frame of reference; the statement of the relativity principle is meaningful; and, moreover, it is satisfied.

78. But, this example obviously represents a special electrodynamic configuration. Indeed, equations (4.7)–(4.8) imply that

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r} - \mathbf{V}t) \tag{4.13}$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0(\mathbf{r} - \mathbf{V}t) \tag{4.14}$$

with some time-independent $\mathbf{E}_0(\mathbf{r})$ and $\mathbf{B}_0(\mathbf{r})$. In other words, the field must be a stationary one, that is, a translation of a static field with velocity **V**. In fact, this corresponds to the very special "rigid" way of persistence we described in point II of the previous section. But, (4.13)–(4.14) is certainly not the case for a general solution of the equations of classical electrodynamics. The behavior of the field can be much more complex. Whatever this complex behavior is, one might hope that it satisfies the general form of persistence described in point III; that is, the equations of persistence are satisfied with a more general local and instantaneous velocity field $\mathbf{v}(\mathbf{r}, t)$:

$$-\frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} = \mathsf{D}\mathbf{E}(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t) \tag{4.15}$$

$$-\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} = \mathsf{D}\mathbf{B}(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t) \tag{4.16}$$

$$-\frac{\partial \varrho(\mathbf{r}, t)}{\partial t} = \nabla\varrho(\mathbf{r}, t) \cdot \mathbf{v}(\mathbf{r}, t) \tag{4.17}$$

In other words, if, as it is usually believed, the relativity principle meaningfully applies to all possible situations in electrodynamics, or, as it is usually believed, the electromagnetic field is a real persisting physical entity, existing in space and time, then there must exist a local instantaneous velocity field $\mathbf{v}(\mathbf{r}, t)$ satisfying (4.15)–(4.16) for all possible solutions of the Maxwell–Lorentz equations (3.35)–(3.39). That is, substituting an arbitrary solution[3] of (3.35)–(3.39) into (4.15)–(4.16), the overdetermined system of equations must have a solution for $\mathbf{v}(\mathbf{r}, t)$.

79. However, one encounters the following difficulty:

Theorem 13. *There exists solution of the coupled Maxwell–Lorentz equations* (3.35)–(3.39) *for which there cannot exist a local instantaneous velocity field* $\mathbf{v}(\mathbf{r}, t)$ *satisfying the persistence equations (4.15)–(4.16).*

Proof. As a proof, we give a surprisingly simple example. Consider the electric field in a parallel-plate capacitor being charged up by a constant current. The electric field strength is:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{u}t \tag{4.18}$$

where $\mathbf{u}$ is a constant vector determined by the current and the properties of the capacitor (Fig. 4.5). It is easy to check that there is no space-time point $(\mathbf{r}, t)$ where $\mathbf{E}(\mathbf{r}, t)$ would satisfy the equation of persistence (4.15) with some velocity $\mathbf{v}(\mathbf{r}, t)$. □

[3] Here we have to be cautious about the existence of solutions to the Maxwell–Lorentz equations (3.35)–(3.39). The content of Paragraph **66** applies to the present case too. Further, we note that since our considerations here focus on the electromagnetic field governed by the four Maxwell equations, we only need to assume the *existence* of a coupled dynamics, approximately described by equations (3.35)–(3.39), and that the coupled equations constitute an initial value problem. In fact, Theorem 14 could be stated in a weaker form, by leaving the concrete form and dynamics of the source densities unspecified.

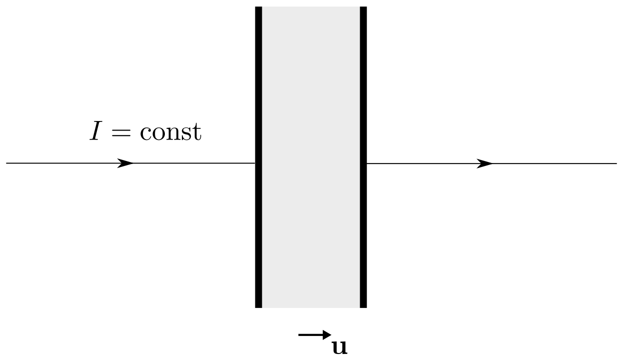

Figure 4.5: Parallel-plate capacitor charged up by a constant current

80. One might think that this is an exceptional case, due to the idealization of the real physical situation. But, as the next theorem shows, this is not so exceptional.

Theorem 14. *There is a dense subset of solutions of the coupled Maxwell–Lorentz equations* (3.35)–(3.39) *for which there cannot exist a local instantaneous velocity field* $\mathbf{v}(\mathbf{r},t)$ *satisfying the persistence equations (4.15)–(4.16).*

Proof. The proof is almost trivial for a locus $(\mathbf{r},t)$ where there is a charged point particle. However, in order to avoid the eventual difficulties concerning the physical interpretation, we are providing a proof for a point $(\mathbf{r}_*,t_*)$ where there is assumed no source at all.

Consider a solution $(\mathbf{r}^1(t), \mathbf{r}^2(t), \ldots, \mathbf{r}^n(t), \mathbf{E}(\mathbf{r},t), \mathbf{B}(\mathbf{r},t))$ of the coupled Maxwell–Lorentz equations (3.35)–(3.39), which satisfies (4.15)–(4.16). At point $(\mathbf{r}_*,t_*)$, the following equations hold:

$$
\begin{aligned}
-\frac{\partial \mathbf{E}(\mathbf{r}_*,t_*)}{\partial t} &= \mathsf{D}\mathbf{E}(\mathbf{r}_*,t_*)\mathbf{v}(\mathbf{r}_*,t_*) && (4.19)\\
-\frac{\partial \mathbf{B}(\mathbf{r}_*,t_*)}{\partial t} &= \mathsf{D}\mathbf{B}(\mathbf{r}_*,t_*)\mathbf{v}(\mathbf{r}_*,t_*) && (4.20)\\
\frac{\partial \mathbf{E}(\mathbf{r}_*,t_*)}{\partial t} &= c^2\nabla\times\mathbf{B}(\mathbf{r}_*,t_*) && (4.21)\\
-\frac{\partial \mathbf{B}(\mathbf{r}_*,t_*)}{\partial t} &= \nabla\times\mathbf{E}(\mathbf{r}_*,t_*) && (4.22)\\
\nabla\cdot\mathbf{E}(\mathbf{r}_*,t_*) &= 0 && (4.23)\\
\nabla\cdot\mathbf{B}(\mathbf{r}_*,t_*) &= 0 && (4.24)
\end{aligned}
$$

Without loss of generality we can assume—at point $\mathbf{r}_*$ and time t_*—that operators $\mathsf{D}\mathbf{E}(\mathbf{r}_*,t_*)$ and $\mathsf{D}\mathbf{B}(\mathbf{r}_*,t_*)$ are invertible and $v_z(\mathbf{r}_*,t_*) \neq 0$.

Now, consider a 3×3 matrix J such that

$$J = \begin{pmatrix} \frac{\partial E_x(\mathbf{r}_*,t_*)}{\partial x} & J_{xy} & J_{xz} \\ \frac{\partial E_y(\mathbf{r}_*,t_*)}{\partial x} & \frac{\partial E_y(\mathbf{r}_*,t_*)}{\partial y} & \frac{\partial E_y(\mathbf{r}_*,t_*)}{\partial z} \\ \frac{\partial E_z(\mathbf{r}_*,t_*)}{\partial x} & \frac{\partial E_z(\mathbf{r}_*,t_*)}{\partial y} & \frac{\partial E_z(\mathbf{r}_*,t_*)}{\partial z} \end{pmatrix} \tag{4.25}$$

with

$$J_{xy} = \frac{\partial E_x(\mathbf{r}_*, t_*)}{\partial y} + \lambda \tag{4.26}$$

$$J_{xz} = \frac{\partial E_x(\mathbf{r}_*, t_*)}{\partial z} - \lambda \frac{v_y(\mathbf{r}_*, t_*)}{v_z(\mathbf{r}_*, t_*)} \tag{4.27}$$

by virtue of which

$$\begin{aligned} J_{xy} v_y(\mathbf{r}_*, t_*) + J_{xz} v_z(\mathbf{r}_*, t_*) &= v_y(\mathbf{r}_*, t_*) \frac{\partial E_x(\mathbf{r}_*, t_*)}{\partial y} \\ &\quad + v_z(\mathbf{r}_*, t_*) \frac{\partial E_x(\mathbf{r}_*, t_*)}{\partial z} \end{aligned} \tag{4.28}$$

Therefore, $J\mathbf{v}(\mathbf{r}_*, t_*) = \mathsf{D}\mathbf{E}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*)$. There always exists a vector field $\mathbf{E}^{\#}_{\lambda}(\mathbf{r})$ such that its Jacobian matrix at point $\mathbf{r}_*$ is equal to J. Obviously, from (4.23) and (4.25), $\nabla \cdot \mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*) = 0$. Therefore, there exists a solution of the Maxwell–Lorentz equations, such that the electric and magnetic fields $\mathbf{E}_{\lambda}(\mathbf{r}, t)$ and $\mathbf{B}_{\lambda}(\mathbf{r}, t)$ satisfy the following conditions:[4]

$$\mathbf{E}_{\lambda}(\mathbf{r}, t_*) = \mathbf{E}^{\#}_{\lambda}(\mathbf{r}) \tag{4.29}$$

$$\mathbf{B}_{\lambda}(\mathbf{r}, t_*) = \mathbf{B}(\mathbf{r}, t_*) \tag{4.30}$$

At $(\mathbf{r}_*, t_*)$, such a solution obviously satisfies the following equations:

$$\frac{\partial \mathbf{E}_{\lambda}(\mathbf{r}_*, t_*)}{\partial t} = c^2 \nabla \times \mathbf{B}(\mathbf{r}_*, t_*) \tag{4.31}$$

$$-\frac{\partial \mathbf{B}_{\lambda}(\mathbf{r}_*, t_*)}{\partial t} = \nabla \times \mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*) \tag{4.32}$$

therefore

$$\frac{\partial \mathbf{E}_{\lambda}(\mathbf{r}_*, t_*)}{\partial t} = \frac{\partial \mathbf{E}(\mathbf{r}_*, t_*)}{\partial t} \tag{4.33}$$

As a little reflection shows, if $\mathsf{D}\mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*)$, that is J, happened to be not invertible,

[4] $\mathbf{E}^{\#}_{\lambda}(\mathbf{r})$ and $\mathbf{B}_{\lambda}(\mathbf{r}, t_*)$ can be regarded as the initial configurations at time t_*; we do not need to specify a particular choice of initial values for the sources.

then one can choose a *smaller* λ such that $\mathsf{D}\mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*)$ becomes invertible (due to the fact that $\mathsf{D}\mathbf{E}(\mathbf{r}_*, t_*)$ is invertible), and, at the same time,

$$\nabla \times \mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*) \neq \nabla \times \mathbf{E}(\mathbf{r}_*, t_*) \tag{4.34}$$

Consequently, from (4.33) , (4.27) and (4.19) we have

$$-\frac{\partial \mathbf{E}_{\lambda}(\mathbf{r}_*, t_*)}{\partial t} = \mathsf{D}\mathbf{E}_{\lambda}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*) = \mathsf{D}\mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*)\mathbf{v}(\mathbf{r}_*, t_*) \tag{4.35}$$

and $\mathbf{v}(\mathbf{r}_*, t_*)$ is uniquely determined by this equation. On the other hand, from (4.32) and (4.34) we have

$$-\frac{\partial \mathbf{B}_{\lambda}(\mathbf{r}_*, t_*)}{\partial t} \neq \mathsf{D}\mathbf{B}_{\lambda}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*) = \mathsf{D}\mathbf{B}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*) \tag{4.36}$$

because $\mathsf{D}\mathbf{B}(\mathbf{r}_*, t_*)$ is invertible, too. That is, for $\mathbf{E}_{\lambda}(\mathbf{r}, t)$ and $\mathbf{B}_{\lambda}(\mathbf{r}, t)$ there is no local and instantaneous velocity at point $\mathbf{r}_*$ and time t_*.

At the same time, λ can be arbitrary small, and

$$\lim_{\lambda \to 0} \mathbf{E}_{\lambda}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \tag{4.37}$$

$$\lim_{\lambda \to 0} \mathbf{B}_{\lambda}(\mathbf{r}, t) = \mathbf{B}(\mathbf{r}, t) \tag{4.38}$$

Therefore solution $\left(\mathbf{r}^1_{\lambda}(t), \mathbf{r}^2_{\lambda}(t), \ldots, \mathbf{r}^n_{\lambda}(t), \mathbf{E}_{\lambda}(\mathbf{r}, t), \mathbf{B}_{\lambda}(\mathbf{r}, t)\right)$ can fall into an arbitrary small neighborhood of $\left(\mathbf{r}^1(t), \mathbf{r}^2(t), \ldots, \mathbf{r}^n(t), \mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t)\right)$.[5] □

4.4 Ontology of Classical Electrodynamics

81. One consequence of these theorems is that the principle of relativity remains an *incomprehensible* statement in the case of a general electrodynamic system. This fact is of course in conflict with the widespread view that the relativity principle is a universal principle valid for all phenomena (cf. Szabó 2004).

The other consequence is also embarrassing: the two *fundamental* electrodynamic quantities, the field strengths $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, do not satisfy the equations

[5] Notice that our investigation has been concerned with the general laws of Maxwell–Lorentz electrodynamics of a coupled particles + electromagnetic field system. The proof of theorem was essentially based on the presumption that all solutions of the Maxwell–Lorentz equations, determined by *any* initial state of the particles + electromagnetic field system, corresponded to physically possible configurations of the electromagnetic field. It is sometimes claimed, however, that the solutions must be restricted by the so called retardation condition, according to which all physically admissible field configurations must be generated from the retarded potentials belonging to some pre-histories of the charged particles (Jánossy 1971, p. 171; Frisch 2005, p. 145). There is no obvious answer to the question of how Theorem 14 is altered under such additional condition.

of persistence (4.6). Therefore, the electromagnetic field individuated by the field strengths cannot be regarded as a persisting physical object; in other words, electromagnetic field—for example, the field within the capacitor in Fig. 4.5—cannot be regarded as being a real physical entity existing in space and time. This seems to contradict to the usual realistic interpretation of classical electrodynamics. So, there are three options.

(i) One can abandon the realist understanding of electrodynamics: There is no such a persisting physical entity as "electromagnetic field".

(ii) Although, we think, in point III we formulated the most general form of how an extended physical object can persist, one may try to imagine a more sophisticated way of persistence.

(iii) Electromagnetic field is a real physical entity, persisting in the sense we formulated persistence in point III, but it cannot be individuated by the field strengths $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. That is, there must exist some quantities other than the field strengths, perhaps outside of the scope of classical electrodynamics, individuating electromagnetic field. This suggests that classical electrodynamics is an ontologically incomplete theory.

82. How to conceive properties, different from the field strengths, which are capable of individuating the electromagnetic field? One might think of them as some "finer", more fundamental, properties of the field, not only individuating it as a persisting extended object, but also determining the values of the field strengths. However, the following easily verifiable theorem shows that this determination cannot be so simple:

Theorem 15. *Let $\{f_i(\mathbf{r},t)\}_{i=1}^n$ be a package of quantities for which there exist a local instantaneous velocity field $\mathbf{v}(\mathbf{r},t)$ satisfying the equations of persistence (4.6) in a given space-time region. If a quantity Φ is a functional of the quantities $f_1, f_2, \ldots, f_n$ in the following form:*

$$\Phi(\mathbf{r},t) = \Phi\left(f_1(\mathbf{r},t), f_2(\mathbf{r},t), ..., f_n(\mathbf{r},t)\right)$$

then Φ also obeys the equation of persistence

$$-\frac{\partial \Phi(\mathbf{r},t)}{\partial t} = \nabla \Phi(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)$$

with the same local instantaneous velocity field $\mathbf{v}(\mathbf{r}, t)$*, within the same space-time region.*

Therefore, $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ cannot supervene pointwise upon some more fundamental individuating quantities satisfying the persistence equations. However, they might supervene in some non-local sense. For example, imagine that $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ provide only a course-grained characterization of the field, but there exist some more fundamental fields $\mathbf{e}(\mathbf{r}, t)$ and $\mathbf{b}(\mathbf{r}, t)$, such that

$$\mathbf{E}(\mathbf{r}, t) = \int\limits_{\Omega} \mathbf{e}\left(\mathbf{r}', t'\right) d^4(\mathbf{r}, t)$$

$$\mathbf{B}(\mathbf{r}, t) = \int\limits_{\Omega} \mathbf{b}\left(\mathbf{r}', t\right) d^4(\mathbf{r}, t)$$

where Ω is a neighborhood of $(\mathbf{r}, t)$. In this case, the more fundamental quantities $\mathbf{e}(\mathbf{r}, t)$ and $\mathbf{b}(\mathbf{r}, t)$ may satisfy the equations of persistence, while $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, supervening on $\mathbf{e}(\mathbf{r}, t)$ and $\mathbf{b}(\mathbf{r}, t)$, may not.

Appendix I

It is assumed that space and time coordinates are defined in all inertial frames of reference; that is, in an arbitrary inertial frame K, space tags $\mathbf{r}(A) = (x(A), y(A), z(A)) \in \mathbb{R}^3$ and a time tag $t(A) \in \mathbb{R}$ are assigned to every event A—by means of some empirical operations. We also assume that the assignment is mutually unambiguous, such that there is a one to one correspondence between the space and time tags in arbitrary two inertial frames of reference K and K'; that is, the tags $(x'(A), y'(A), z'(A))$ can be expressed by the tags $(x(A), y(A), z(A))$, and vice versa. The concrete form of this functional relation is an empirical question. In this paper, we take it for granted that this functional relation is the well-known Lorentz transformation

Below we recall the most important formulas we use. For the sake of simplicity, we assume the usual situation: K' is moving along the x-axis with velocity $\mathbf{V} = (V, 0, 0)$ relative to K, the corresponding axises are parallel and the two origins coincide at time 0.

The connection between the space and time tags of an event A in K and K' is the following:

$$x'(A) = \frac{x(A) - Vt(A)}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{4.39}$$

$$y'(A) = y(A) \tag{4.40}$$

$$z'(A) = z(A) \tag{4.41}$$

$$t'(A) = \frac{t(A) - \frac{V}{c^2}x(A)}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{4.42}$$

Let A be an event on the worldline of a particle. For the velocity of the particle at A we have:

$$v'_x(A) = \frac{v_x(A) - V}{1 - \frac{v_x(A)V}{c^2}} \tag{4.43}$$

$$v'_y(A) = \frac{v_y(A)\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{v_x(A)V}{c^2}} \tag{4.44}$$

$$v'_z(A) = \frac{v_z(A)\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{v_x(A)V}{c^2}} \tag{4.45}$$

We also use the inverse transformation in the following special case:

$$\mathbf{v}'(A) = (v', 0, 0) \quad \mapsto \quad \mathbf{v}(A) = \left(\frac{v' + V}{1 + \frac{v'V}{c^2}}, 0, 0\right) \tag{4.46}$$

$$\mathbf{v}'(A) = (0, 0, v') \quad \mapsto \quad \mathbf{v}(A) = \left(V, 0, v'\sqrt{1 - \frac{V^2}{c^2}}\right) \tag{4.47}$$

The transformation rule of acceleration is much more complex, but we need it only for $\mathbf{v}'(A) = (0, 0, 0)$:

$$a'_x(A) = \frac{a_x(A)}{\left(1 - \frac{V^2}{c^2}\right)^{\frac{3}{2}}} \tag{4.48}$$

$$a'_y(A) = \frac{a_y(A)}{1 - \frac{V^2}{c^2}} \tag{4.49}$$

$$a'_z(A) = \frac{a_z(A)}{1 - \frac{V^2}{c^2}} \tag{4.50}$$

We will also need the y-component of acceleration in case of $\mathbf{v}'(A) = (0, 0, v')$:

$$a'_y(A) = \frac{a_y(A)}{1 - \frac{V^2}{c^2}} \tag{4.51}$$

Appendix II

There are two major versions of the textbook derivation of the transformation rules for electrodynamic quantities from the hypothesis of covariance. The first version follows Einstein's 1905 paper:

(1a) The transformation rules of electric and magnetic field strengths are derived from the presumption of the covariance of the homogeneous (with no sources) Maxwell equations.

(1b) The transformation rules of source densities are derived from the transformations of the field variables.

(1c) From the transformation rules of charge and current densities, it is derived that electric charge is an invariant scalar.

The second version is this:

(2a) The transformation rules of the charge and current densities are derived from some additional *assumptions*; typically from one of the followings:

- (2a1) the invariance of electric charge (Jackson 1999, pp. 553–558)
- (2a2) the current density is of form $\varrho\mathbf{u}(\mathbf{r},t)$, where $\mathbf{u}(\mathbf{r},t)$ is a velocity field (Tolman 1949, p. 85; Møller 1955, p. 140).

(2b) The transformation of the field strengths are derived from the transformation of ϱ and $\mathbf{j}$ and from the presumption of the covariance of the inhomogeneous Maxwell equations.

Unfortunately, with the only exception of (1b), none of the above steps is completely correct. Without entering into the details, let us mention that (2a1) and (2a2) both involve some further empirical information about the world, which does not follow from the simple assumption of covariance. Even in case of (1a) we must have the tacit assumption that zero charge and current densities go to zero charge and current densities during the transformation—otherwise the covariance of the homogeneous

Maxwell equations would not follow from the assumed covariance of the Maxwell equations.

One encounters the next major difficulty in both (1a) and (2b): neither the homogeneous nor the inhomogeneous Maxwell equations determine the transformation rules of the field variables uniquely; $\mathbf{E}'$ and $\mathbf{B}'$ are only determined by $\mathbf{E}$ and $\mathbf{B}$ up to an arbitrary solution of the homogeneous equations (see also Huang 2008).

Finally, let us mention a conceptual confusion that seems to be routinely overlooked in (1c), (2a1) and (2a2). There is no such thing as a simple relation between the scalar invariance of charge and the transformation of charge and current densities, as is usually claimed. For example, it is meaningless to say that

$$Q = \varrho \Delta W = Q' = \varrho' \Delta W' \tag{4.52}$$

where ΔW denotes a volume element, and

$$\Delta W' = \gamma \Delta W \tag{4.53}$$

Whose charge is Q, which remains invariant? Whose volume is ΔW and in what sense is that volume Lorentz contracted? In another form, in (2a2), whose velocity is $\mathbf{u}(\mathbf{r}, t)$?

References

Arntzenius, F. (2000): Are There Really Instantaneous Velocities? *The Monist* **83**, 187–208.

Arthur J. W. (2011): Understanding Geometric Algebra for Electromagnetic Theory (IEEE Press Series on Electromagnetic Wave Theory), Wiley-IEEE Press, Hoboken, NJ.

Bandyopadhyay, B. (2012): *Physics of Nanostructured Solid State Devices.* New York, Springer.

Bell, J. S. (1987): How to teach special relativity, in *Speakable and unspeakable in quantum mechanics.* Cambridge, Cambridge University Press.

Bell, J. S. (1992): George Francis FitzGerald, *Physics World* **5**, 31.

Bohm, D. (1996): *Special Theory of Relativity*, London and New York, Routledge.

Brading, K. and Brown, H. R. (2004): Are Gauge Symmetry Transformations Observable? *British Journal for the Philosophy of Science* **55**, 645.

Brading, K. and Castellani, E. (2008): Symmetry and Symmetry Breaking, *The Stanford Encyclopedia of Philosophy (Fall 2008 Edition)*, E. N. Zalta (ed.), http://plato.stanford.edu/archives/fall2008/entries/symmetry-breaking.

Brown, H. R. (2005): *Physical relativity – space-time structure from a dynamical perspective.* Oxford, New York, Oxford University Press.

Butterfield, J. (2005): On the Persistence of Particles, *Foundations of Physics* **35**, 233–269.

Butterfield, J. 2006: The Rotating Discs Argument Defeated, *British Journal for the Philosophy of Science* **57**, 1–45.

Comstock, D. F. (1909): The principle of relativity, *Science* **31**, 767.

Dieks, D. (2006): Another Look at General Covariance and the Equivalence of Reference Frames, *Studies In History and Philosophy of Modern Physics* **37**, 174.

Einstein, A (1905): On the Electrodynamics of Moving Bodies, in H. A. Lorentz et al., *The principle of relativity: a collection of original memoirs on the special and general theory of relativity.* London, Methuen and Company, 1923

Einstein, A (1916): The Foundations of the General Theory of Relativity, in H. A. Lorentz et al., *The principle of relativity: a collection of original memoirs on the special and general theory of relativity.* London, Methuen and Company, 1923.

Einstein, A (1940): The Fundamentals of Theoretical Physics, *Ideas and Opinions*, New York, Bonanza.

Earman, J. and Norton, J. D. (1987): What Price Spacetime Substantivalism? The Hole Story, *British Journal for the Philosophy of Science* **38** , 515–525.

Earman, J. (2004): Laws, Symmetry, and Symmetry Breaking: Invariance, Conservation Principles, and Objectivity, *Philosophy of Science* **71**, 1227.

Feynman, R., Leighton, R.B. and Sands, M. (1964): *The Feynman Lectures on Physics, Vol. II: Mainly Electromagnetism and Matter*, Reading, MA, Addison-Wesley.

Fock, V. (1964): *The theory of space, time and gravitation* (2nd Revised Edition). Oxford, Pergamon Press.

Friedman, M. (1983): *Foundations of space-time theories.* Princeton NJ, Princeton University Press.

Frisch, M. (2005): *Inconsistency, Asymmetry, and Non-Locality*, Oxford: Oxford University Press.

Galilei, Galileo (1953): *Dialogue concerning the Two Chief World Systems, Ptolemaic & Copernican*, Berkeley: University of California Press.

Georgiou, A. (1969): Special relativity and thermodynamics, *Proc. Comb. Phil. Soc.* **66**, 423.

Giuliani, G. (2008): A General Law for Electromagnetic Induction, *Europhysics Letters* **81**, 1–6.

Gömöri, M. and Szabó, L. E. (2015): Formal Statement of the Special Principle of Relativity, *Synthese* **192** (7), 2053–2076, DOI: 10.1007/s11229-013-0374-1

Gömöri, M. and Szabó, L. E. (2014): How to move an electromagnetic field?, (http://philsci-archive.pitt.edu/10766/).

Gömöri, M. and Szabó, L. E. (2013): Operational understanding of the covariance of classical electrodynamics, *Physics Essays* **26**, 361.

Griffiths, D.J. (1999): *Introduction to electrodynamics.* Prentice Hall, Upper Saddle River NJ.

Grøn, Ø. and Vøyenli, K. (1999): On the Foundation of the Principle of Relativity, *Foundations of Physics* **29**, 1695.

Hawley, K. (2001): *How Things Persist*, Oxford: Oxford University Press.

Hestenes D. (1966): *Space-Time Algebra,* New York, Gordon & Breach.

Hestenes, D. (2003): Spacetime physics with geometric algebra, *Am. J. Phys.* **71**, 691,

Houtappel, R. M. F, Van Dam, H. and Wigner, E. P. (1965): The Conceptual Basis and Use of the Geometric Invariance Principles, *Reviews of Modern Physics* **37**, 595.

Huang, Young-Sea (1993): Has the Lorentz-covariant electromagnetic force law been directly tested experimentally?, *Foundations of Physics Letters* **6**, 257.

Huang, Young-Sea (2008): Does the manifestly covariant equation $\partial_\alpha \mathbf{A}^\alpha = 0$ imply that $\mathbf{A}^\alpha$ is a four-vector?, *Canadian J. Physics* **86**, pp. 699–701 DOI: 10.1139/P08-012.

Huang, Young-Sea (2009): A new perspective on relativistic transformation for Maxwell's equations of electrodynamics, *Physica Scripta* **79**, 055001 (5pp) DOI: 10.1088/0031-8949/79/05/055001.

Ivezić, T. (2001): "True Transformations Relativity" and Electrodynamics, *Foundations of Physics* **31**, 1139.

Ivezić, T. (2003): The Proof that the Standard Transformations of E and B Are not the Lorentz Transformations, *Foundations of Physics* **33**, 1339.

Jackson, J. D. (1962): *Classical Electrodynamics (First edition).* New York, John Wiley & Sons.

Jackson, J. D. (1999): *Classical Electrodynamics (Third edition).* Hoboken (NJ), John Wiley & Sons.

Jammer, M. (2000): *Concepts of Mass in Contemporary Physics and Philosophy.* Princeton, Princeton University Press.

Jánossy, L. (1971): *Theory of relativity based on physical reality.* Budapest, Akadémiai Kiadó.

Landau, L. D. and Lifshitz, E. M. (1980), *The Classical Theory of Fields*, Oxford, Butterworth–Heinemann.

Madarász, J. X. (2002): *Logic and Relativity (in the light of definability theory).* PhD thesis, Eötvös University, Budapest, 2002, http://www.math-inst.hu/pub/algebraic-logic/Contents.html.

Møller C. (1955): *The Theory of Relativity.* Oxford, Clarendon Press.

Muller, F. (2007): Inconsistency in Classical Electrodynamics?, *Philosophy of Science* **74**, pp. 253–277.

Norton, J. D. (1988): Coordinates and Covariance: Einstein's view of spacetime and the modern view, *Foundations of Physics* **19**, 1215.

Norton, J. D. (1993): General Covariance and the Foundations of General Relativity: Eight Decades of Dispute, *Reports on Progress in Physics* **56**, 791.

Norton, J. D. (2004), Einstein's Investigations of Galilean Covariant Electrodynamics prior to 1905, *Archive for History of Exact Sciences* **59**, 45–105.

Norton, J. D. (2013): Special Theory of Relativity: The Principles, http://www.pitt.edu/~jdnorton/teaching/HPS_0410/chapters/Special_relativity_principles

Pauli, W. (1958): *Theory of Relativity*, London, Pergamon.

Poincaré H. (1956): The Principles of Mathematical Physics, *The Scientific Monthly* **82**, 165.

Reichenbach, H. (1965): *The Theory of Relativity and A Priori Knowledge.* Berkeley and Los Angeles, University of California Press.

Rindler, W. (2006): *Relativity: Special, General, and Cosmological* (2nd Edition), Oxford, Oxford University Press.

Rohrlich, F. (2007): *Classical Charged Particles.* Singapore, World Scientific.

Rovelli, C. (2004): *Quantum Gravity*, Cambridge, Cambridge University Press, Sections 2.2–2.3

Sardesai, P. L. (2004): *A Primer of Special Relativity*, New Delhi, Verlag: New Age International.

Scanlon, P. J., Henriksen, R. N. and Allen, J. R. (1969): Approaches to Electromagnetic Induction, *Am. J. Phys.* **37**, 698–708.

Schlick, M. (1920): *Space and Time in Contemporary Physics: An Introduction to the Theory of Relativity and Gravitation*, New York, Oxford University Press (republished: Mineola, N.Y., Dower Publications, Inc., 2005).

Sewell, G. L. (2008): On the question of temperature transformations under Lorentz and Galilei boosts, *Journal of Physics A: Mathematical and Theoretical* **41**, 382003.

Sider, Theodore 2001: *Four-Dimensionalism*, Oxford: Oxford University Press.

Szabó, L. E. (2004): On the meaning of Lorentz covariance, *Foundations of Physics Letters* **17**, 479.

Szabó, L. E. (2010): Empirical Foundation of Space and Time, in M. Suárez, M. Dorato and M. Rédei (eds.), *EPSA07: Launch of the European Philosophy of Science Association*, Springer, 251–266.

Szabó, L. E. (2011): Lorentzian theories vs. Einsteinian special relativity – a logico-empiricist reconstruction, in A. Máté, M. Rédei and F. Stadler (eds.), *Vienna Circle and Hungary – Veröffentlichungen des Instituts Wiener Krei*s, Springer

Tolman, R. C. (1917): *The theory of relativity of motion.* Berkeley, University of California Press.

Tolman, R. C. (1949): *Relativity, Thermodynamics and Cosmology.* Oxford, Clarendon Press.

Tooley, M. (1988): In Defence of the Existence of States of Motion, *Philosophical Topics* 16:225–254.

Westman, H. and Sonego, S. (2009): Coordinates, observables and symmetry in relativity, *Ann. Phys.* **324**, 1585.

Zimmerman Jones, A. and Robbins, D. (2009): *String Theory For Dummies*, Hoboken, NJ., Wiley Publishing, Inc.